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THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SECOND SERIES

Volume 4 *No.* 15 *September* 1953

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PRINTED IN GREAT BRITAIN BY CHARLES BATEY AT THE UNIVERSITY PRESS, OXFORD

THE QUARTERLY JOURNAL OF
MATHEMATICS

OXFORD SECOND SERIES

Edited by T. W. CHAUNDY, U. S. HASLAM-JONES,
J. H. C. THOMPSON

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ON INTEGRAL ANALOGUES OF CERTAIN TRANSFORMATIONS OF WELL-POISED BASIC HYPERGEOMETRIC SERIES

By R. P. AGARWAL (*London*)

[Received 26 June 1952]

1. Introduction

As long ago as 1909 Watson (**5**) used basic integrals of Barnes's type to obtain transformations of general basic series. Recently Sears (**3**) used a method not involving integrals to obtain transformations of well-poised basic series of any order, and Slater (**4**) has more recently used special kinds of basic integrals (not of Barnes's type) to prove these results. Sears also used his methods to obtain transformations of well-poised basic series corresponding to the transformations of series of the ordinary type given in Chapter VI of Bailey's tract.† The object of this paper is to develop a method of obtaining some of the latter results by means of integrals. The integrals used are similar to those used by Watson.

In § 3 I give the basic analogue of Barnes's second lemma‡ and in § 4 it is used to obtain a basic integral representing well-poised series. In § 5 the integral analogue of basic Dougall's theorem is given.

2. Notation

I shall use the following notation throughout this paper. Assuming $|q| < 1$ let

$$(a)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}), \qquad (a)_0 = 1,$$

$$(a)_{-n} = (-)^n q^{\frac{1}{2}n(n+1)}/a^n (q/a)_n,$$

$${}_r\Phi_s\left[\begin{matrix} q^{a_1}, q^{a_2}, \ldots, q^{a_r}; \\ q^{b_1}, q^{b_2}, \ldots, q^{b_s} \end{matrix} q^x\right] = \sum_{n=0}^{\infty} \frac{(q^{a_1})_n \ldots (q^{a_r})_n}{(q)_n (q^{b_1})_n \ldots (q^{b_s})_n} q^{nx}.$$

The last series will be denoted by

$${}_r\mathscr{F}_s\left[\begin{matrix} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s \end{matrix} x\right].$$

Whenever there is a parameter of the type $-q^{a_m}$ in ${}_r\Phi_s$, it will be represented by $(a_m)^*$ in ${}_r\mathscr{F}_s$.

† Bailey: Cambridge Tract, *Generalized Hypergeometric Series* (1935). I shall henceforth refer to it as 'Tract'.

‡ Tract, § 6.2 (1).

For well-poised series with special forms of second and third parameters, i.e. for

$$ {}_r\Phi_s\left[\begin{matrix} q^a, q^{1+\frac{1}{2}a}, -q^{1+\frac{1}{2}a}, q^b, q^c,\ldots; \\ q^{\frac{1}{2}a}, -q^{\frac{1}{2}a}, q^{1+a-b}, q^{1+a-c},\ldots \end{matrix} q^{2+2a-b-c\ldots}\right], $$

I shall write

$$ {}_r\Omega_s[a; b, c,\ldots]. $$

Also,

$$ \wp\left[\begin{matrix} a_1, a_2,\ldots, a_r; \\ b_1, b_2,\ldots, b_s \end{matrix}\right] = \prod_{n=0}^{\infty} \frac{(1-q^{a_1+n})\ldots(1-q^{a_r+n})}{(1-q^{b_1+n})\ldots(1-q^{b_s+n})}, $$

and, whenever there is a factor of the type $(1+q^{a_m+n})$ in the product, I shall denote it by $(a_m)^*$ in the product $\wp$.

I shall also put

$$ \log q = -\omega = -(\omega_1+i\omega_2), $$

where ω, ω_1, ω_2 are definite numbers, ω_1 and ω_2 being real and, since $|q| < 1$, $\omega_1 > 0$.

In any integral, in the paper, the contour of integration C is a line parallel to the line $\mathrm{re}(\omega s) = 0$, with loops, if necessary, to ensure that the poles forming an increasing sequence lie to the right of the contour while the poles forming a decreasing sequence lie to the left of the contour.

3. The basic analogue of Barnes's second lemma

Watson† has proved the following basic analogue of Barnes's first lemma‡

$$ \frac{1}{2\pi i}\int_C \wp\left[\begin{matrix} 1-\xi+s, 1-\eta+s; \\ \alpha_1+s, \quad \alpha_2+s \end{matrix}\right] \frac{\pi q^{s^2}\,ds}{\sin\pi(\xi-s)\sin\pi(\eta-s)} $$

$$ = \frac{q^{\xi+\eta}}{\sin\pi(\xi-\eta)(q^{\xi}-q^{\eta})}\wp\left[\begin{matrix} 1, \xi-\eta, \eta-\xi, \alpha_1+\alpha_2+\xi+\eta; \\ \alpha_1+\xi, \alpha_1+\eta, \alpha_2+\xi, \alpha_2+\eta \end{matrix}\right]. \quad (3.1) $$

Let $\xi = n$ (a positive integer) and $\eta = \gamma-\alpha_1-\alpha_2$ in (3.1); then we get, after some simplification,

$$ \frac{1}{2\pi i}\int_C \wp\left[\begin{matrix} 1-n+s, 1-\gamma+\alpha_1+\alpha_2+s; \\ \alpha_1+s, \quad \alpha_2+s \end{matrix}\right] \frac{\pi q^{s^2}\,ds}{\sin\pi s\sin\pi(\gamma-\alpha_1-\alpha_2-s)} $$

$$ = \operatorname{cosec}\pi(\alpha_1+\alpha_2-\gamma)\wp\left[\begin{matrix} 1+\alpha_1+\alpha_2-\gamma, \gamma-\alpha_1-\alpha_2, \gamma, 1; \\ \alpha_1, \alpha_2, \gamma-\alpha_1, \gamma-\alpha_2 \end{matrix}\right] \times $$

$$ \times \frac{(-)^n(q^{\alpha_1})_n(q^{\alpha_2})_n}{(q^{\gamma})_n} q^{-\frac{1}{2}n(n-1)+n(\gamma-\alpha_1-\alpha_2)}. \quad (3.2) $$

† (5) 286, Lemma B. The notation has been changed by using an equivalent expression in infinite products instead of Jackson's basic gamma functions.

‡ See Bailey's Tract, p. 6.

Now, multiplying both sides of (3.2) by $(q^c)_n q^{n(\beta_2-c)}/(q)_n(q^{\beta_2})_n$ and summing for n from zero to infinity, we get, on reduction and change of order of integration and summation (a process easily justifiable),

$$\operatorname{cosec}\pi(\alpha_1+\alpha_2-\gamma)\wp\begin{bmatrix}1+\alpha_1+\alpha_2-\gamma,\ \gamma-\alpha_1-\alpha_2,\ \gamma,\ 1;\\ \alpha_1,\ \alpha_2,\ \gamma-\alpha_1,\ \gamma-\alpha_2\end{bmatrix}\times$$

$$\times\,{}_3\mathscr{F}_2\begin{bmatrix}\alpha_1,\ \alpha_2,\ c;\\ \gamma,\ \beta_2\end{bmatrix}\gamma+\beta_2-\alpha_1-\alpha_2-c\Big]$$

$$=\frac{1}{2\pi i}\int_C \wp\begin{bmatrix}1+s,\ 1-\gamma+\alpha_1+\alpha_2+s;\\ \alpha_1+s,\qquad \alpha_2+s\end{bmatrix}\frac{\pi q^s}{\sin\pi s\sin\pi(\gamma-\alpha_1-\alpha_2-s)}\times$$

$$\times\,{}_2\mathscr{F}_1\begin{bmatrix}-s,\ c;\\ \beta_2\end{bmatrix}s+\beta_2-c\Big]\,ds. \quad (3.3)$$

Summing the ${}_2\mathscr{F}_1$ on the right by the basic analogue of Gauss's theorem [Tract 8.4 (3)], we get

$$\operatorname{cosec}\pi(\alpha_1+\alpha_2-\gamma)\wp\begin{bmatrix}1+\alpha_1+\alpha_2-\gamma,\ \gamma-\alpha_1-\alpha_2,\ \gamma,\ \beta_2,\ 1;\\ \alpha_1,\ \alpha_2,\ \gamma-\alpha_1,\ \gamma-\alpha_2,\ \beta_2-c\end{bmatrix}\times$$

$$\times\,{}_3\mathscr{F}_2\begin{bmatrix}\alpha_1,\ \alpha_2,\ c;\\ \gamma,\ \beta_2\end{bmatrix}\gamma+\beta_2-\alpha_1-\alpha_2-c\Big]$$

$$=\frac{1}{2\pi i}\int_C \wp\begin{bmatrix}1+s,\ \beta_2+s,\ 1-\gamma+\alpha_1+\alpha_2+s;\\ \alpha_1+s,\ \alpha_2+s,\qquad \beta_2-c+s\end{bmatrix}\frac{\pi q^s\,ds}{\sin\pi s\sin\pi(\gamma-\alpha_1-\alpha_2-s)}. \quad (3.4)$$

Now take $\gamma=c$; the series on the left can be summed by the basic analogue of Gauss's theorem and we get, on putting

$$\beta_1=1-c+\alpha_1+\alpha_2, \qquad \alpha_3=\beta_2-c,$$

i.e.
$$\beta_1+\beta_2=1+\alpha_1+\alpha_2+\alpha_3,$$

that

$$\frac{1}{2\pi i}\int_C \wp\begin{bmatrix}1+s,\ \beta_1+s,\ \beta_2+s;\\ \alpha_1+s,\ \alpha_2+s,\ \alpha_3+s\end{bmatrix}\frac{\pi q^s\,ds}{\sin\pi s\sin\pi(1-\beta_1-s)}$$

$$=\operatorname{cosec}\pi(\beta_1-1)\wp\begin{bmatrix}1,\ \beta_1,\ 1-\beta_1,\ \beta_2-\alpha_1,\ \beta_2-\alpha_2,\ \beta_2-\alpha_3;\\ \alpha_1,\ \alpha_2,\ \alpha_3,\ 1+\alpha_1-\beta_1,\ 1+\alpha_2-\beta_1,\ 1+\alpha_3-\beta_1\end{bmatrix} \quad (3.5)$$

provided that
$$\beta_1+\beta_2=1+\alpha_1+\alpha_2+\alpha_3,$$

the integral being convergent for $\operatorname{re}[s\log q-\log\sin\pi s\sin\pi(\beta_1+s)]<0$, when $|s|$ is large. This gives the basic analogue of Barnes's second lemma.

4. Integrals representing basic well-poised series

It is easily verified by (3.5) that

$$\operatorname{cosec}\pi(\beta_1-1)\,\wp\left[\begin{matrix}1,\ \beta_1+n,\ 1-\beta_1-n,\ k-\alpha_1+n,\ k-\alpha_2+n,\ k-\alpha_3+n;\\ \alpha_1+n,\ \alpha_2+n,\ \alpha_3+n,\ k-\alpha_2-\alpha_3,\ k-\alpha_1-\alpha_2,\ k-\alpha_1-\alpha_3\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_C \wp\left[\begin{matrix}k+s,\ 1-k+\alpha_1+\alpha_2+\alpha_3+s,\ 1+s;\\ \alpha_1+s,\ \alpha_2+s,\ \alpha_3+s\end{matrix}\right]\times$$

$$\times\frac{(-1)^n\pi q^{s(n+1)-\frac{1}{2}n(n+1)}(q^{-s})_n}{\sin\pi s\sin\pi(1-\beta_1-s)(q^{k+s})_n}\,ds,$$

where $\beta_1=1-k+\alpha_1+\alpha_2+\alpha_3$.

Simplifying the left-hand side we see, by expansion and the interchange of the order of integration and summation, that

$$\operatorname{cosec}\pi(\beta_1-1)\,\wp\left[\begin{matrix}1,\ \beta_1,\ 1-\beta_1,\ k-\alpha_1,\ k-\alpha_2,\ k-\alpha_3;\\ \alpha_1,\ \alpha_2,\ \alpha_3,\ k-\alpha_1-\alpha_2,\ k-\alpha_1-\alpha_3,\ k-\alpha_2-\alpha_3\end{matrix}\right]\times$$

$$\times{}_{r+4}\mathscr{F}_{r+3}\left[\begin{matrix}a,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \rho_1,\ldots,\ \rho_r;\\ k-\alpha_1,\ k-\alpha_2,\ k-\alpha_3,\ \sigma_1,\ldots,\ \sigma_r\end{matrix}\ \eta-\beta_1\right]$$

$$=\frac{1}{2\pi i}\int_C \wp\left[\begin{matrix}k+s,\ 1-k+\alpha_1+\alpha_2+\alpha_3+s,\ 1+s;\\ \alpha_1+s,\ \alpha_2+s,\ \alpha_3+s\end{matrix}\right]\frac{\pi q^s}{\sin\pi s\sin\pi(1-\beta_1-s)}\times$$

$$\times{}_{r+2}\mathscr{F}_{r+1}\left[\begin{matrix}a,\ \rho_1,\ldots,\ \rho_r,\ -s;\\ \sigma_1,\ldots,\ \sigma_r,\ k+s\end{matrix}\ \eta+s-1\right]ds \qquad (4.1)$$

where $\beta_1=1-k+\alpha_1+\alpha_2+\alpha_3$ and η is an arbitrary parameter.

Thus, if we can sum the series on the right of (4.1) in terms of infinite products of the above type, we can find an integral representing the series on the left. As an illustration take

$$r=4,\quad k=1+a,\quad \rho_1=1+\tfrac{1}{2}a,\quad \rho_2=(1+\tfrac{1}{2}a)^*,\quad \rho_3=b,\quad \rho_4=c,$$

$$\sigma_1=\tfrac{1}{2}a,\quad \sigma_2=(\tfrac{1}{2}a)^*,\quad \sigma_3=1+a-b,\quad \sigma_4=1+a-c,\quad \eta=2+a-b-c.$$

On the right we get a series ${}_6\mathscr{F}_5$ which can be summed:

$${}_6\mathscr{F}_5\left[\begin{matrix}a,\ 1+\tfrac{1}{2}a,\ (1+\tfrac{1}{2}a)^*,\ b,\ c,\ d;\\ \tfrac{1}{2}a,\ (\tfrac{1}{2}a)^*,\ 1+a-b,\ 1+a-c,\ 1+a-d\end{matrix}\ 1+a-b-c-d\right]$$

$$=\wp\left[\begin{matrix}1+a,\ 1+a-b-c,\ 1+a-b-d,\ 1+a-c-d;\\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-b-c-d\end{matrix}\right],$$

which is a particular case of Tract § 8.5 (3), where $1+a=d+e$.

We thus obtain an integral representing a well-poised ${}_8\mathscr{F}_7$, namely

$$
\begin{aligned}
{}_8\Omega_7[a;\alpha_1,\alpha_2,\alpha_3,b,c] &= \sin\pi(\alpha_1+\alpha_2+\alpha_3-a-1)\times \\
&\times \wp\left[\begin{matrix}1+a,\alpha_1,\alpha_2,\alpha_3,1+a-\alpha_1-\alpha_2,1+a-\alpha_2-\alpha_3,1+a-\alpha_1-\alpha_3,1+a-b-c;\\ 1,\beta_1,1-\beta_1,1+a-\alpha_1,1+a-\alpha_2,1+a-\alpha_3,1+a-b,1+a-c\end{matrix}\right]\times \\
&\times \frac{1}{2\pi i}\int_C \wp\left[\begin{matrix}\alpha_1+\alpha_2+\alpha_3-a+s,1+a-b+s,1+a-c+s,1+s;\\ \alpha_1+s,\alpha_2+s,\alpha_3+s,1+a-b-c+s\end{matrix}\right]\frac{\pi q^s\,ds}{\sin\pi s\sin\pi(1-\beta_1-s)},
\end{aligned}
\tag{4.2}
$$

where $\beta_1 = \alpha_1+\alpha_2+\alpha_3-a$.

If, however, following Watson, we evaluate the integral on the right of (4.2) by considering the residues at the poles of

$$\operatorname{cosec}\pi s \quad \text{and} \quad \operatorname{cosec}\pi(1-\beta_1-s)$$

lying to the right of the contour C, we get the well-known relation between an ${}_8\mathscr{F}_7$ and two Saalschützian ${}_4\mathscr{F}_3$.†

As another illustration let us replace a by k, b by $b+k-a$, c by $c+k-a$ and α_1 by α_1+k-a and take $k+\alpha_1+b+c = 1+2a$ in (4.2). Then the right-hand integral in (4.2) remains unchanged, and we get a relation between two well-poised ${}_8\mathscr{F}_7$ series, a transformation given by Bailey‡ (**1**). It gives an exact analogue of a relation between two well-poised ${}_7F_6$ given in Tract § 7.5 (1).

5. An integral analogue of basic Dougall's theorem

In (4.2), replacing α_1, α_2, α_3 by d, e, f respectively we get, on imposing the additional restriction on the parameters

$$1+2a = b+c+d+e+f,$$

that

$$
\begin{aligned}
&\omega(a;b,c,d,e,f) \\
&= \sin\pi(d+e+f-a-1)\,\wp\left[\begin{matrix}1+a-d-e,1+a-e-f,1+a-d-f;\\ 1,1-\beta,b,c\end{matrix}\right]\times \\
&\quad\times\frac{1}{2\pi i}\int_C \wp\left[\begin{matrix}1+a-b+s,1+a-c+s,1+s;\\ d+s,\quad e+s,\quad f+s\end{matrix}\right]\frac{\pi q^s\,ds}{\sin\pi s\sin\pi(1-\beta-s)},
\end{aligned}
\tag{5.1}
$$

where $\beta = d+e+f-a$ and

$$
\begin{aligned}
&\omega(a;b,c,d,e,f) \\
&= \wp\left[\begin{matrix}1+a-d,1+a-e,1+a-f,1+a-b,1+a-c;\\ 1+a,b,c,d,e,f\end{matrix}\right]{}_8\Omega_7[a;b,c,d,e,f].
\end{aligned}
$$

† Tract § 8.5 (3), after changing q^a to a and so on.

‡ Bailey's transformation ((**1**) 4.3) is obtained after changing q^a to a and so on in the relation between two ${}_8\mathscr{F}_7$.

By analogy we get

$$\omega[2b-a;\, b,\, b+c-a,\, b+d-a,\, b+e-a,\, b+f-a]$$
$$= -\sin \pi c\, \wp\begin{bmatrix} 1+a-d-e,\ 1+a-e-f,\ 1+a-d-f; \\ 1,\ 1-\beta,\ b,\ c \end{bmatrix} \times$$
$$\times \frac{1}{2\pi i} \int_C \wp\begin{bmatrix} 1+b-a+s,\ 1+b-c+s,\ 1+s; \\ b+d-a+s,\ b+e-a+s,\ b+f-a+s \end{bmatrix} \frac{\pi q^s\, ds}{\sin \pi s \sin \pi(c-s)} \tag{5.2}$$

provided that $1+2a = b+c+d+e+f$.

Putting $s+b-a = t$ in (5.2) so that the products in the integrand in (5.2) become the same as in (5.1) and adding to (5.1) we get, after slightly simplifying the integrand on the right, that

$$\omega(a;\, b,\, c,\, d,\, e,\, f) -$$
$$-q^{b-a}\omega(2b-a;\, b,\, b+c-a,\, b+d-a,\, b+e-a,\, b+f-a)$$
$$= \wp\begin{bmatrix} 1+a-d-e,\ 1+a-e-f,\ 1+a-d-f; \\ 1,\ b,\ c,\ 1-\beta \end{bmatrix}$$
$$\times \frac{1}{2\pi i} \int_C \wp\begin{bmatrix} 1+a-b+s,\ 1+a-c+s,\ 1+s; \\ d+s,\ e+s,\ f+s \end{bmatrix} \frac{\pi q^s \sin \pi(a-b)\, ds}{\sin \pi s \sin \pi(b-a-s)} \tag{5.3}$$

provided that $1+2a = b+c+d+e+f$.

Now, following Watson, it can be very easily verified, by considering the residues at the poles of the integrand that lie to the right of C, that the left-hand expression in (5.3) is equal to $\{\wp(1)\}^{-1}$ times the integral

$$\frac{1}{2\pi i} \int_C \wp\begin{bmatrix} 1+s,\ \tfrac{1}{2}a+s,\ (\tfrac{1}{2}a+s)^*,\ 1+a-b+s,\ 1+a-c+s, \\ a+s,\ 1+\tfrac{1}{2}a+s,\ (1+\tfrac{1}{2}a+s)^*,\ b+s,\ c+s, \\ 1+a-d+s,\ 1+a-e+s,\ 1+a-f+s; \\ d+s,\ e+s,\ f+s \end{bmatrix} \frac{\pi q^s \sin \pi(a-b)\, ds}{\sin \pi s \sin \pi(b-a-s)}. \tag{5.4}$$

Equating (5.4) and the right-hand integral of (5.3) we get

$$\frac{1}{2\pi i} \int_C \wp\begin{bmatrix} 1+s,\ \tfrac{1}{2}a+s,\ (\tfrac{1}{2}a+s)^*,\ 1+a-b+s,\ 1+a-c+s, \\ a+s,\ 1+\tfrac{1}{2}a+s,\ (1+\tfrac{1}{2}a+s)^*,\ b+s,\ c+s, \\ 1+a-d+s,\ 1+a-e+s,\ 1+a-f+s; \\ d+s,\ e+s,\ f+s \end{bmatrix} \frac{\pi q^s\, ds}{\sin \pi s \sin \pi(b-a-s)}$$
$$= \wp\begin{bmatrix} 1+a-d-e,\ 1+a-e-f,\ 1+a-d-f \\ b,\ c,\ 1-\beta \end{bmatrix} \times$$
$$\times \frac{1}{2\pi i} \int_C \wp\begin{bmatrix} 1+a-b+s,\ 1+a-c+s,\ 1+s; \\ d+s,\ e+s,\ f+s \end{bmatrix} \frac{\pi q^s\, ds}{\sin \pi s \sin \pi(b-a-s)}, \tag{5.5}$$

provided that $1+2a = b+c+d+e+f$, where $\beta = d+e+f-a$.

Now, the right-hand integral of (5.5) can be evaluated by the help of (3.5) under the condition $1+2a = b+c+d+e+f$, and hence we obtain the result

$$\frac{1}{2\pi i}\int_C \wp\begin{bmatrix}1+s, & \frac{1}{2}a+s, & (\frac{1}{2}a+s)^*, & 1+a-b+s, & 1+a-c+s,\\ a+s, & 1+\frac{1}{2}a+s, & (1+\frac{1}{2}a+s)^*, & b+s, & c+s,\end{bmatrix}$$
$$\begin{matrix}1+a-d+s, & 1+a-e+s, & 1+a-f+s;\\ d+s, & e+s, & f+s\end{matrix}\Bigg]\frac{\pi q^s\,ds}{\sin\pi s\sin\pi(b-a-s)}$$
$$= \operatorname{cosec}\pi(a-b)\,\wp\begin{bmatrix}1, & 1+a-b, & b-a, & 1+a-d-e, & 1+a-e-f,\\ b, & c, & d, & e, & f,\end{bmatrix}$$
$$\begin{matrix}1+a-f-d, & 1+a-c-d, & 1+a-c-e, & 1+a-c-f;\\ b+c-a, & d+b-a, & e+b-a, & f+b-a\end{matrix}\Bigg] \tag{5.6}$$

provided that $1+2a = b+c+d+e+f$. This gives us the integral analogue of the basic Dougall's theorem.

If, however, we evaluate the integral on the left of (5.6) by considering the poles of $\operatorname{cosec}\pi s$ and $\operatorname{cosec}\pi(b-a-s)$ that lie to the right of the contour C, we get a relation between two well-poised ${}_8\Phi_7$ series given by Bailey (**2**) in 1947. This gives the form which the basic analogue of Dougall's theorem assumes when we remove the restriction that one of the numerator parameters must be a negative integer. It can be easily shown that the integral converges when

$$\operatorname{re}[s\log q - \log\sin\pi s\sin\pi(b-a-s)] < 0 \quad \text{for } |s| \text{ large.}$$

I am thankful to Professor Bailey for suggesting the problem and for his kind suggestions leading to the preparation of this paper.

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NOTE ON SOME PARTITION FORMULAE

By L. CARLITZ (*Duke University*)

[Received 25 November 1952]

1. In a recent paper (**1**), Bailey showed that Ramanujan's identity

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5\prod_{n=1}^{\infty}\frac{(1-x^{5n})^5}{(1-x^n)^6}, \tag{1.1}$$

can be derived from the familiar formula

$$\wp(u)-\wp(v) = -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)},$$

when the $\wp$-functions are replaced by their Fourier series. It seems natural to apply the same idea to the formula

$$\wp'(u) = -\frac{\sigma(2u)}{\sigma^4(u)}. \tag{1.2}$$

Since

$$\wp'(u) = -2\left(\frac{\pi}{2\omega_1}\right)^3\frac{\cos(\pi u/2\omega_1)}{\sin^3(\pi u/2\omega_1)} + 2\left(\frac{\pi}{\omega_1}\right)^3\sum_1^{\infty}\frac{m^2q^{2m}}{1-q^{2m}}\sin\frac{m\pi u}{\omega_1},$$

and

$$\sigma(u) = \frac{2\omega_1}{\pi}\exp\left(\frac{\eta_1 u^2}{2\omega_1}\right)\sin\frac{\pi u}{2\omega_1}\prod_{n=1}^{\infty}\frac{\{1-q^{2n}\exp(\pi u/\omega_1)\}\{1-q^{2n}\exp(-\pi u/\omega_1)\}}{(1-q^{2n})^2},$$

(1.2) becomes

$$1+a^{-1}\frac{(1-a)^3}{1+a}\sum_1^{\infty}\frac{n^2q^{2n}}{1-q^{2n}}(a^n-a^{-n})$$

$$= \prod_1^{\infty}\frac{(1-q^{2n}a^2)(1-q^{2n}a^{-2})(1-q^{2n})^6}{(1-q^{2n}a)^4(1-q^{2n}a^{-1})^4}, \tag{1.3}$$

where $a = \exp(\pi iu/\omega_1)$. This formula can also be deduced from (5) of (**1**) by dividing by $a-b$ and then letting $b \to a$.

2. If we put $q^2 = x^3$ and $a = x$, (1.3) becomes

$$1+x^{-1}\frac{(1-x)^3}{1+x}\sum_1^\infty \frac{n^2x^{3n}}{1-x^{3n}}(x^n-x^{-n})$$

$$= \prod_1^\infty \frac{(1-x^{3n+2})(1-x^{3n-2})(1-x^{3n})^6}{(1-x^{3n+1})^4(1-x^{3n-1})^4}$$

$$= \frac{(1-x)^4}{1-x^2}\prod_1^\infty \frac{(1-x^{3n})^6}{(1-x^{3n-1})^3(1-x^{3n-2})^3}$$

$$= \frac{(1-x)^3}{1+x}\prod_1^\infty \frac{(1-x^{3n})^9}{(1-x^n)^3},$$

which may be written

$$x\prod_1^\infty \frac{(1-x^{3n})^9}{(1-x^n)^3} = \frac{x(1+x)}{(1-x)^3} + \sum_1^\infty \frac{n^2x^{3n}}{1-x^{3n}}(x^n+x^{-n}). \tag{2.1}$$

It is easily verified that

$$\frac{x(1+x)}{(1-x)^3} = \sum_1^\infty n^2x^n.$$

Thus the terms in the right-hand member of (2.1) with exponents divisible by 3 contribute

$$9\sum_1^\infty n^2x^{3n} + 9\sum \frac{n^2x^{9n}}{1-x^{9n}}(x^{3n}+x^{-3n}).$$

Hence, if we define $p_k(m)$ by means of

$$\prod_{n=1}^\infty (1-x^n)^{-k} = \sum_{m=0}^\infty p_k(m)x^m,$$

we see that (2.1) implies

$$\sum_1^\infty p_3(3n-1)x^{3n}\prod_1^\infty (1-x^{3n})^9 = 9x^3\prod_1^\infty \frac{(1-x^{9n})^9}{(1-x^{3n})^3}$$

and therefore

$$\sum_0^\infty p_3(3m+2)x^m = 9\prod_1^\infty \frac{(1-x^{3n})^9}{(1-x^n)^{12}}. \tag{2.2}$$

3. If we put $q^2 = x$, $a = \omega = \exp(\frac{2}{3}i\pi)$ in (1.3), we get

$$1+3\omega(1-\omega)\sum_1^\infty \frac{n^2x^n}{1-x^n}(\omega^n-\omega^{-n})$$

$$= \prod_1^\infty \frac{(1-x^n)^6}{(1-x^n\omega)^3(1-x^n\omega^2)^3}$$

$$= \prod_1^\infty \frac{(1-x^n)^9}{(1-x^{3n})^3}.$$

Now $\omega^n-\omega^{-n}=0$ when $3\mid n$, and

$$\omega(1-\omega)(\omega^n-\omega^{-n})=\begin{cases}-3 & n\equiv 1 \pmod 3,\\ 3 & n\equiv 2 \pmod 3.\end{cases}$$

Hence we get the identity

$$1-9\sum_1^\infty \left(\frac{n}{3}\right)\frac{n^2x^n}{1-x^n}=\prod_1^\infty \frac{(1-x^n)^9}{(1-x^{3n})^3}, \tag{3.1}$$

where $(n/3)$ denotes the Legendre–Jacobi symbol.

The terms in the left member of (3.1) with exponents divisible by 3 evidently contribute

$$1-9\sum_1^\infty \left(\frac{n}{3}\right)\frac{n^2x^{3n}}{1-x^{3n}}=\prod_1^\infty \frac{(1-x^{3n})^9}{(1-x^{9n})^3}.$$

We thus get the formula

$$\sum_0^\infty p_{-9}(3m)x^m=\prod_1^\infty \frac{(1-x^n)^{12}}{(1-x^{3n})^3}. \tag{3.2}$$

4. In exactly the same way, if we take $q^2=x^4$, $a=x$ in (1.3), we get, after a little reduction,

$$\frac{x(1+x)}{(1-x)^3}+\sum_1^\infty \frac{n^2x^{4n}}{1-x^{4n}}(x^n-x^{-n})=x\prod_1^\infty \frac{(1-x^{2n})^6(1-x^{4n})^4}{(1-x^n)^4}. \tag{4.1}$$

From (4.1) we get, for example,

$$\sum_0^\infty p_4(2m+1)x^m=4\prod_1^\infty \frac{(1-x^{2n})^2(1-x^{4n})^4}{(1-x^n)^{10}}. \tag{4.2}$$

A similar identity is obtained by retaining only exponents divisible by 4 in (4.1).

In the next place, if we take $q^2=x$, $a=i$ in (1.3), the result is

$$1-4\sum_1^\infty \left(\frac{-4}{n}\right)\frac{n^2x^n}{1-x^n}=\prod_1^\infty \frac{(1-x^{2n})^6(1-x^n)^4}{(1-x^{4n})^4}. \tag{4.3}$$

From (4.3) follows

$$\sum_0^\infty p_{-4}(2m)x^m=\prod_1^\infty \frac{(1-x^{2n})^{10}}{(1-x^n)^2(1-x^{4n})^4}. \tag{4.4}$$

5. Similar but more complicated results are obtained by taking $q^2 = x^r$, $a = x$ in (1.3). For example, for $r = 5$, we get after a little simplification

$$\frac{x(1+x)}{(1-x)^3} + \sum_1^\infty \frac{n^2x^{5n}}{1-x^{5n}}(x^n+x^{-n}) = x\prod_1^\infty \frac{(1-x^n)(1-x^{5n})^5}{(1-x^{5n-1})^5(1-x^{5n-4})^5}. \tag{5.1}$$

If we put $$\prod_1^\infty \frac{1-x^n}{(1-x^{5n-1})^5(1-x^{5n-4})^5} = \sum_0^\infty q(m)x^m,$$

then (5.1) implies

$$\sum_0^\infty q(5m+4)x^m = 25\prod_1^\infty \frac{(1-x^{5n})^5}{(1-x^n)^4(1-x^{5n-1})^5(1-x^{5n-4})^5}. \tag{5.2}$$

In this connexion it may be of interest to observe that Ramanujan's formula

$$\prod_1^\infty \frac{(1-x^n)^5}{1-x^{5n}} = 1-5\sum_1^\infty \left(\frac{n}{5}\right)\frac{nx^n}{1-x^n}, \tag{5.3}$$

which is proved in (**1**), implies

$$\sum_0^\infty p_{-5}(5m)x^m = \prod_1^\infty (1-x^n)^6(1-x^{5n})^{-1}; \tag{5.4}$$

indeed it is only necessary to consider the terms in (5.3) with exponent divisible by 5. The identity (5.4) is stated in a recent paper by Newman [(**2**) 320], who points out that it had been found earlier (but not published) by D. H. Lehmer.

Comparison of (5.4) with (1.1) leads to a rather curious result:

$$\sum_0^\infty p_{-5}(5m)x^m \sum_0^\infty p(5m+4)x^m = 5\prod_1^\infty (1-x^{5n})^4,$$

so that $$\sum_{r=0}^m p_{-5}(5m-5r)p(5r+4) = \begin{cases} 0 & (5 \nmid m), \\ 5p_{-4}(k) & (m = 5k). \end{cases}$$

6. Finally we notice that, if we take $q = x^8$, $a = x$, $b = x^3$ in formula (5) of (**1**), we get

$$\sum_1^\infty \left(\frac{2}{n}\right)\frac{x^n}{(1-x^n)^2} = x\prod_1^\infty \frac{(1-x^{2n})^3(1-x^{4n})(1-x^{3n})^2}{(1-x^n)^2}, \tag{6.1}$$

while, if $q = x$, $a = \exp(2\pi i/8)$, $b = \exp(6\pi i/8)$, we get

$$\prod_1^\infty \frac{(1-x^n)^2(1-x^{2n})(1-x^{4n})^3}{(1-x^{8n})^2} = 1-2\sum_1^\infty \left(\frac{2}{n}\right)\frac{nx^n}{1-x^n}. \qquad (6.2)$$

It seems scarcely worth while writing down the partition formulae implied by (6.1) and (6.2).

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APPLICATIONS OF SOME BASIC IDENTITIES

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[Received 25 November 1952]

1. As is well known, Gauss (**4**) deduced from the identity

$$\sum_{s=0}^{2m}(-1)^s\begin{bmatrix}2m\\ s\end{bmatrix}=\prod_{r=1}^{m}(1-q^{2r-1}), \tag{1.1}$$

where

$$\begin{bmatrix}m\\ s\end{bmatrix}=\frac{(1-q^m)(1-q^{m-1})\ldots(1-q^{m-s+1})}{(1-q)(1-q^2)\ldots(1-q^s)}, \qquad \begin{bmatrix}m\\ 0\end{bmatrix}=1,$$

the formula

$$\sum_{s=0}^{n-1}\epsilon^{s^2}=\prod_{r=1}^{\frac{1}{2}(n-1)}(\epsilon^{2r-1}-\epsilon^{-(2r-1)}), \tag{1.2}$$

where n is odd and ϵ denotes a primitive nth root of unity. It may be remarked that (1.2) is valid not only in the complex field but in any field containing n distinct nth roots of unity. Thus, for example, the formula holds in the finite field $GF(p^t)$, where $p^t\equiv 1 \pmod n$; in particular for $t=1$, $p\equiv 1 \pmod n$, we may interpret (1.2) as a congruence $(\mathrm{mod}\, p)$. In this special case we remark that the sum in (1.2) satisfies

$$\Big(\sum_{s=0}^{n-1}\epsilon^{s^2}\Big)^2\equiv(-1)^{\frac{1}{2}(n-1)}n \pmod p. \tag{1.3}$$

In view of the above it may be of interest to derive a few more formulae similar to (1.2) from known basic identities. In each case several interpretations of the final formulae are possible.

2. In the familiar identity

$$\prod_{r=1}^{m-1}(1+q^rx)=\sum_{r=0}^{m-1}\begin{bmatrix}m-1\\ r\end{bmatrix}q^{\frac{1}{2}r(r+1)}x^r, \tag{2.1}$$

replace q by α, a primitive $2m$th root of unity. Since $\alpha^m=-1$, we have

$$(1-\alpha^{m-1})\ldots(1-\alpha^{m-r})=(1+\alpha)\ldots(1+\alpha^r)\alpha^{-\frac{1}{2}r(r+1)},$$

and (2.1) reduces to

$$\prod_{r=1}^{m-1}(1+\alpha^rx)=\sum_{r=0}^{m-1}\frac{(1+\alpha)\ldots(1+\alpha^r)}{(1-\alpha)\ldots(1-\alpha^r)}x^r. \tag{2.2}$$

Here again we may think of α as a complex number or alternatively as a number of $GF(p^t)$, where $p^t\equiv 1 \pmod{2m}$.

In exactly the same way, if in (1.1) we replace q by α, a primitive $(4m+2)$th root of unity, we get after a little manipulation

$$\sum_{r=0}^{2m} \frac{(1+\alpha)\dots(1+\alpha^r)}{(1-\alpha)\dots(1-\alpha^r)} \alpha^{\frac{1}{2}r(r+1)} = \prod_{r=1}^{m} (1+\alpha^{2r}). \tag{2.3}$$

In particular, (2.3) implies

$$\sum_{r=0}^{\frac{1}{2}(p-3)} \frac{(1+g)\dots(1+g^r)}{(1-g)\dots(1-g^r)} g^{\frac{1}{2}r(r+1)} \equiv \prod_{r=1}^{\frac{1}{2}(p-1)} (1+g^{2r}) \pmod{p}, \tag{2.4}$$

where $p \equiv 3 \pmod 4$ and g is a primitive root $(\bmod p)$. If $p \equiv 1 \pmod 4$, the sum in (2.4) is divisible by p.

We may also note that the left-hand member of (1.1) can be written as

$$(-1)^m \begin{bmatrix} 2m \\ m \end{bmatrix} + 2 \sum_{s=1}^{m} (-1)^{m+s} \begin{bmatrix} 2m \\ m+s \end{bmatrix};$$

this leads to the formula

$$1+2\sum_{s=1}^{m} (-1)^s \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-s+1})}{(1-q^{m+1})(1-q^{m+2})\dots(1-q^{m+s})} = (-1)^m \frac{(1-q)\dots(1-q^m)}{(1+q)\dots(1+q^m)}. \tag{2.5}$$

Hence, if $q = \alpha^4$, where α is a primitive $(4m+1)$th root of unity, (2.5) becomes after a little manipulation

$$1+2\sum_{s=1}^{m} a^{-s(2s-1)} \frac{(1-\alpha)(1-\alpha^5)\dots(1-\alpha^{4s-3})}{(1-\alpha^3)(1-\alpha^7)\dots(1-\alpha^{4s-1})} = \frac{(1-\alpha)(1-\alpha^5)\dots(1-\alpha^{4m-3})}{(1+\alpha)(1+\alpha^5)\dots(1+\alpha^{4m-3})}, \tag{2.6}$$

while, when $q = \alpha^4$, where α is a primitive $(4m+3)$th root of unity, we get

$$1+2\sum_{s=1}^{m} \alpha^{-s(2s+1)} \frac{(1-\alpha^3)(1-\alpha^7)\dots(1-\alpha^{4s-1})}{(1-\alpha)(1-\alpha^5)\dots(1-\alpha^{4s-3})} = \frac{(1-\alpha^3)(1-\alpha^7)\dots(1-\alpha^{4m-1})}{(1+\alpha^3)(1+\alpha^7)\dots(1+\alpha^{4m-1})}. \tag{2.7}$$

3. In (2.1) replace x by $-x$. Multiplying corresponding members of the two formulae we easily obtain†

$$\sum_{r=0}^{2m} (-1)^r q^{r(r-2m)} \begin{bmatrix} 2m \\ r \end{bmatrix}^2 = (-1)^m q^{-m^2} \begin{bmatrix} 2m \\ m \end{bmatrix}', \tag{3.1}$$

where

$$\begin{bmatrix} 2m \\ m \end{bmatrix}' = \frac{(1-q^{4m})(1-q^{4m-2})\dots(1-q^{2m+2})}{(1-q^2)(1-q^4)\dots(1-q^{2m})}.$$

† (3.1) is a particular case of the basic analogue of Kummer's theorem, which can be deduced from (5.1) below.

Put $q = \alpha$, a primitive $(4m+2)$th root of unity. We find that

$$\sum_{r=0}^{2m} \left\{\frac{(1+\alpha)\ldots(1+\alpha^r)}{(1-\alpha)\ldots(1-\alpha^r)}\right\}^2 = (-1)^m. \tag{3.2}$$

In the next place, exactly as in the proof of (2.5), we can transform (3.1) into

$$1+2\sum_{1}^{m}(-1)^r q^{r^2}\left\{\frac{(1-q^m)\ldots(1-q^{m-r+1})}{(1-q^{m+1})\ldots(1-q^{m+r})}\right\}^2 = \begin{bmatrix}2m\\ m\end{bmatrix}'\begin{bmatrix}2m\\ m\end{bmatrix}^{-2}. \tag{3.3}$$

Then, when $q = \alpha^4$, a primitive $(4m+1)$th root of unity, (3.3) becomes

$$1+2\sum_{1}^{m}(-1)^r\alpha^{2r}\left\{\frac{(1-\alpha)(1-\alpha^5)\ldots(1-\alpha^{4r-3})}{(1-\alpha^3)(1-\alpha^7)\ldots(1-\alpha^{4r-1})}\right\}^2 = \prod_{1}^{m}\frac{1+\alpha^{4r-1}}{1-\alpha^{4r-1}}\cdot\frac{1-\alpha^{4r}}{1+\alpha^{4r}}; \tag{3.4}$$

when $q = \alpha^4$, a primitive $(4m+3)$th root of unity we get

$$1+2\sum_{1}^{m}(-1)^r\alpha^{-2r}\left\{\frac{(1-\alpha^3)(1-\alpha^7)\ldots(1-\alpha^{4r-1})}{(1-\alpha)(1-\alpha^5)\ldots(1-\alpha^{4r-3})}\right\}^2 = \prod_{1}^{m}\frac{1+\alpha^{4r-3}}{1-\alpha^{4r-3}}\cdot\frac{1-\alpha^{4r}}{1+\alpha^{4r}}. \tag{3.5}$$

Formulae (3.4) and (3.5) may be compared with (2.6) and (2.7). It may also be of interest to compare them with the following formulae due to Oltramare [(3) 277]

$$1+(m!)^4 \equiv -2\left\{\left(\frac{1}{3}\right)^2+\left(\frac{1.5}{3.7}\right)^2+\left(\frac{1.5.9}{3.7.11}\right)^2+\ldots\right\} \pmod{p = 4m+1},$$

$$2^5+(m!)^4 \equiv -2^6\left\{\left(\frac{3}{1}\right)^2+\left(\frac{3.7}{1.5}\right)^2+\left(\frac{3.7.11}{1.5.9}\right)^2+\ldots\right\} \pmod{p = 4m+3},$$

where the moduli are primes.

4. In the present connexion Jackson's formula† [(2) 175]

$$\sum_{-m}^{m}(-1)^r\begin{bmatrix}2m\\ m+r\end{bmatrix}^3 q^{\frac{1}{2}r(3r+1)} = \frac{(q)_{3m}}{\{(q)_m\}^3} \tag{4.1}$$

is of interest. In the first place we replace (4.1) by

$$1+\sum_{1}^{m}(-1)^r(q^{\frac{1}{2}r(3r+1)}+q^{\frac{1}{2}r(3r-1)})\left\{\frac{(1-q^m)\ldots(1-q^{m-r+1})}{(1-q^{m+1})\ldots(1-q^{m+r})}\right\}^3 = \frac{(q)_{3m}}{\{(q^{m+1})_m\}^3}. \tag{4.2}$$

† (4.1) can also be deduced from (5.1). See W. N. Bailey, *Quart. J. of Math.* (Oxford) (2) 1 (1950), 318–20.

Then exactly as before, if we take $q = \alpha^4$, a primitive $(4m+1)$th root of unity, (4.2) becomes

$$1+\sum_1^m \alpha^r(1+\alpha^{4r})\left(\frac{(1-\alpha)(1-\alpha^5)\dots(1-\alpha^{4r-3})}{(1-\alpha^3)(1-\alpha^7)\dots(1-\alpha^{4r-1})}\right)^3 = \prod_1^m \frac{(1-\alpha^{4r})(1-\alpha^{4r-2})}{(1-\alpha^{4r-1})^2}, \tag{4.3}$$

while, when $q = \alpha^4$, a primitive $(4m+3)$th root of unity, we get

$$1+\sum_1^m \alpha^{-r}(1-\alpha^{-4r})\left(\frac{(1-\alpha^3)(1-\alpha^7)\dots(1-\alpha^{4r-1})}{(1-\alpha)(1-\alpha^5)\dots(1-\alpha^{4r-3})}\right)^3 = \prod_1^m \frac{(1-\alpha^{4r})(1-\alpha^{4r-6})}{(1-\alpha^{4r-3})^2}. \tag{4.4}$$

On the other hand, when $q = \alpha^{-1}$, a primitive $(4m+2)$th root of unity, (4.1) yields without much trouble

$$\sum_0^{2m}(-1)^r\alpha^{rm}\left\{\frac{(1+\alpha)\dots(1+\alpha^r)}{(1-\alpha)\dots(1-\alpha^r)}\right\}^3 = \frac{2}{1+\alpha^m}\prod_1^m\left(\frac{1+\alpha^r}{1-\alpha^r}\right)^2. \tag{4.5}$$

5. Finally in Jackson's analogue of Dougall's theorem [(**1**) 67]†

$${}_8\Phi_7\left[\begin{matrix}a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-m};\\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{m+1}, & q\end{matrix}\right] = \frac{(aq)_m(aq/cd)_m(aq/bd)_m(aq/bc)_m}{(aq/b)_m(aq/c)_m(aq/d)_m(aq/bcd)_m}, \tag{5.1}$$

where $bcde = a^2q^{m+1}$, let

$$a \to 1, \qquad b = c = d = e, \qquad b^4 = q^{m+1}.$$

Then the left-hand member of (5.1) becomes

$$1+\sum_1^m (1+q^r)q^r\frac{(q^{\frac{1}{4}(m+1)})_r^4(q^{-m})_r}{(q^{\frac{1}{4}(3-m)})_r^4(q^{m+1})_r}. \tag{5.2}$$

If we suppose m even and take $q = \alpha^5$, a primitive $(5m+1)$th root of unity, this reduces to

$$1+\sum_1^m (1+\alpha^{5r})\alpha^{5r}\left\{\frac{(1-\alpha)(1-\alpha^6)\dots(1-\alpha^{5r-4})}{(1-\alpha^4)(1-\alpha^9)\dots(1-\alpha^{5r-1})}\right\}^5.$$

On the other hand the right-hand member of (5.1) reduces first to

$$\frac{(q)_m(q^{\frac{1}{4}(1-m)})_m^3}{(q^{\frac{1}{4}(3-m)})_m^3(q^{\frac{1}{4}(1-3m)})_m},$$

and finally to

$$(-\alpha)^m\frac{(1-\alpha^5)\dots(1-\alpha^{5m})\cdot\{(1-\alpha^3)\dots(1-\alpha^{5m-2})\}^3}{\{(1-\alpha^4)\dots(1-\alpha^{5m-1})\}^4}.$$

† This gives (2.5), the alternative form of (1.1), if we substitute for z and let $d \to \infty$, and then let $a \to 1$, $c \to 0$ and take $b = -1$.

We therefore get (replacing m by $2m$)

$$1+\sum_{1}^{2m}(1+\alpha^{5r})\alpha^{5r}\left\{\frac{(1-\alpha)(1-\alpha^{6})\ldots(1-\alpha^{5r-4})}{(1-\alpha^{4})(1-\alpha^{9})\ldots(1-\alpha^{5r-1})}\right\}^{5}$$

$$=\alpha^{2m}\prod_{1}^{2m}\frac{(1-\alpha^{5r})(1-\alpha^{5r-2})^{3}}{(1-\alpha^{5r-1})^{4}}, \qquad (5.3)$$

where α is a primitive $(10m+1)$th root of unity.

In the same way it follows from (5.2) that

$$1+\sum_{1}^{4m}(1+\alpha^{8r})\alpha^{8r}\left\{\frac{(1-\alpha)(1-\alpha^{9})\ldots(1-\alpha^{8r-7})}{(1-\alpha^{7})(1-\alpha^{15})\ldots(1-\alpha^{8r-1})}\right\}^{4}$$

$$=\prod_{1}^{4m}\frac{(1-\alpha^{8r})(1-\alpha^{8r-2})^{3}}{(1-\alpha^{8r-1})^{4}}, \qquad (5.4)$$

where α is a primitive $(8m+1)$th root of unity.

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ON THE NUMBER OF POINTS OF A GIVEN LATTICE IN A RANDOM HYPERSPHERE

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[Received 16 December 1952]

1. In an earlier paper (**4**) one of us considered the number of points of a square lattice which lie (i) in a randomly located circle of given radius, and (ii) in a randomly located oval of given size and shape. The results obtained in these two problems have since been extended by Hlawka (**2**), who has considered the k-dimensional analogue of (ii) in which the randomly located oval is replaced by a randomly located k-dimensional convex body. In this paper we propose to vary the problem (i) in another direction; we shall be concerned with a randomly located k-dimensional hypersphere in relation to a given k-dimensional lattice and shall consider the behaviour of the expectation of the number of points of the lattice in the hypersphere and its variance. For the latter we obtain two expressions, one of which involves an infinite series of Bessel functions, as in (**4**), while the other is of finite closed form; both are closely related to the positive definite quadratic form (and its adjugate) associated with the lattice.

We consider in especial detail the particular case when the lattice is the plane hexagonal lattice (i.e. the pattern formed by the vertices of a pack of equilateral triangles). The reason for considering this particular lattice is that the results are of interest in connexion with Mr. Gordon Wright's theory (**8**) of cutaneous sensation and it was at his suggestion that the investigation was undertaken.

2. Let Λ be a given lattice in k-dimensional Euclidean space R_x, defined by the equations

$$x_i = \sum_{j=1}^{k} a_{ij} y_j \quad (i = 1, 2, \ldots, k), \tag{1}$$

where the y_j ($j = 1, 2, \ldots, k$) run through all integers, and where

$$\Delta = \det a_{ij} > 0.$$

The relations (1) may be written in matrix form as $\mathbf{x} = \mathbf{ay}$, where $\mathbf{x}$ and $\mathbf{y}$ denote column vectors with components x_i and y_j ($i, j = 1, 2, \ldots, k$) respectively, and $\mathbf{a}$ is the matrix $\{a_{ij}\}$. In addition to defining the lattice, we also regard the equation $\mathbf{x} = \mathbf{ay}$ as defining a one-to-one mapping

between points $\mathbf{x}$ of R_x and points $\mathbf{y}$ of another k-dimensional space R_y.

If dashes denote transposes, we have

$$\sum_{i=1}^{k} x_i^2 = \mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{b}\mathbf{y} = q(\mathbf{y}), \tag{2}$$

say, where $\mathbf{b} = \mathbf{a}'\mathbf{a}$ and $q(\mathbf{y})$ is a positive definite quadratic form in the real variables $y_1, y_2, \ldots, y_k$. If $\mathbf{A}$ and $\mathbf{B}$ are the adjugate matrices to $\mathbf{a}$ and $\mathbf{b}$ respectively, then $\mathbf{B} = \mathbf{A}\mathbf{A}'$, and the adjugate form is

$$Q(\mathbf{y}) = \mathbf{y}'\mathbf{B}\mathbf{y}. \tag{3}$$

The determinants of the two forms q and Q are Δ^2 and Δ^{2k-2} respectively.

Now let $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be any point of R_x and denote by $l(\rho, \boldsymbol{\alpha})$ the number of points of Λ in the hypersphere

$$\sum_{i=1}^{k} (x_i - \alpha_i)^2 \leqslant \rho^2. \tag{4}$$

Let $\boldsymbol{\beta}$ be the image of $\boldsymbol{\alpha}$ in R_y, so that $\boldsymbol{\alpha} = \mathbf{a}\boldsymbol{\beta}$, and let Λ_0 be the lattice in R_y which consists of all points $\mathbf{y}$ with integral components. Then the points of the hypersphere (4) correspond to points $\mathbf{y}$ of the hyperellipsoid

$$q(\mathbf{y} - \boldsymbol{\beta}) \leqslant \rho^2,$$

and $l(\rho, \boldsymbol{\alpha}) = L(\rho, \boldsymbol{\beta})$, say, is the number of points of Λ_0 which lie in this hyperellipsoid.

Then $L(\rho, \boldsymbol{\beta})$ is a periodic function of period unity in the k components of $\boldsymbol{\beta}$. It can be represented, exactly as in (4), by a k-fold Fourier series

$$L(\rho, \boldsymbol{\beta}) \sim \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \ldots \sum_{m_k=-\infty}^{\infty} a(m_1, m_2, \ldots, m_k) \times \\ \times \exp 2\pi i (m_1\beta_1 + m_2\beta_2 + \ldots + m_k\beta_k),$$

which we can write more briefly as

$$L(\rho, \boldsymbol{\beta}) \sim \sum_{\mathbf{m} \in \Lambda_0} a(\mathbf{m}) e^{2\pi i \mathbf{m}'\boldsymbol{\beta}}. \tag{5}$$

Just as in (4) or (2) we find that the coefficients $a(\mathbf{m})$ are given by†

$$\begin{aligned} a(\mathbf{m}) &= \int_0^1 d\beta_1 \int_0^1 d\beta_2 \ldots \int_0^1 L(\rho, \boldsymbol{\beta}) e^{-2\pi i \mathbf{m}'\boldsymbol{\beta}}\, d\beta_k \\ &= \iint\limits_{q(\mathbf{y}) \leqslant \rho^2} \ldots \int e^{2\pi i \mathbf{m}'\mathbf{y}}\, dy \\ &= \frac{1}{\Delta} \iint\limits_{\mathbf{x}'\mathbf{x} \leqslant \rho^2} \ldots \int e^{2\pi i \mathbf{m}'\mathbf{a}^{-1}\mathbf{x}}\, dx \\ &= \frac{\rho^{\frac{1}{2}k}}{\Delta \omega^{\frac{1}{2}k}} J_{\frac{1}{2}k}(2\pi\rho\omega), \end{aligned}$$

† We write dy for $dy_1\, dy_2 \ldots dy_k$ etc.

when $\mathbf{m} \neq \mathbf{o} = \{0, 0, ..., 0\}$, where the positive number ω is defined by $\omega^2 = \mathbf{m}'\mathbf{a}^{-1}(\mathbf{a}^{-1})'\mathbf{m} = Q(\mathbf{m})/\Delta^2$. Thus

$$a(\mathbf{m}) = \frac{\Delta^{\frac{1}{2}k-1}\rho^{\frac{1}{2}k}}{\{Q(\mathbf{m})\}^{\frac{1}{4}k}} J_{\frac{1}{2}k}\left\{\frac{2\pi\rho}{\Delta}\sqrt{Q(\mathbf{m})}\right\} \quad (\mathbf{m} \neq \mathbf{o}), \tag{6}$$

and it is easily seen that

$$a(\mathbf{o}) = K_k \rho^k/\Delta, \tag{7}$$

where

$$K_k = \pi^{\frac{1}{2}k}/\Gamma(\tfrac{1}{2}k+1) \tag{8}$$

is the content of the unit hypersphere in k dimensions.

3. One can now proceed exactly as in (**4**) or (**2**), the basic probability assumption being that the centre of the hypersphere is to be uniformly distributed within one of the fundamental cells of the original lattice Λ. The coefficient $a(\mathbf{o})$ gives the *expected number of points of* Λ *in the hypersphere* and so

$$\mathscr{E}(l) = \frac{K_k \rho^k}{\Delta} = \frac{(\pi\rho^2)^{\frac{1}{2}k}}{\Delta\Gamma(\frac{1}{2}k+1)}. \tag{9}$$

Thus $\mathscr{E}(l)$ is equal to the ratio of the content of the hypersphere to the content of a fundamental cell of the lattice, a result which could have been anticipated by other arguments. Of more interest is the *variance of the number of points of* Λ *in the hypersphere*; an application of Parseval's theorem shows this to be given by

$$\sigma^2 = \{\sigma(\rho, \Lambda)\}^2 = \sum_{\mathbf{m}}{}' |a(\mathbf{m})|^2$$

$$= \Delta^{k-2}\rho^k \sum_{\mathbf{m}}{}' \left\{J_{\frac{1}{2}k}\left(\frac{2\pi\rho}{\Delta}\sqrt{Q(\mathbf{m})}\right)\right\}^2 \Big/ \{Q(\mathbf{m})\}^{\frac{1}{2}k}, \tag{10}$$

by (6), where the dash denotes a summation over all $\mathbf{m} \in \Lambda_0$ other than $\mathbf{m} = \mathbf{o}$.

This series may be used to evaluate σ^2 for all ρ, although it is only when ρ is large that it is likely to be of practical use. For large ρ we can replace the Bessel function by the first term of its asymptotic expansion and so obtain

$$\sigma^2 = A(\rho)\rho^{k-1} + O(\rho^{k-2}), \tag{11}$$

where the coefficient $A(\rho)$ is defined by

$$A(\rho) = \frac{\Delta^{k-1}}{\pi^2} \sum_{\mathbf{m}}{}' \cos^2\left[\frac{2\pi\rho}{\Delta}\sqrt{\{Q(\mathbf{m})\}} - \tfrac{1}{4}(k+1)\pi\right] \Big/ \{Q(\mathbf{m})\}^{\frac{1}{2}(k+1)}. \tag{12}$$

If we write

$$\bar{A} = \lim_{R\to\infty} \frac{1}{R} \int_0^R A(\rho)\, d\rho \tag{13}$$

for the average value of the almost-periodic function $A(\rho)$, we find that

$$\bar{A} = \lim_{R\to\infty} \frac{1}{R} \int_0^R \frac{\sigma^2\, d\rho}{\rho^{k-1}} = \frac{\Delta^{k-1}}{2\pi^2} \sum_{\mathbf{m}}{}' \{Q(\mathbf{m})\}^{-\frac{1}{2}(k+1)} \tag{14}$$

$$= \frac{\Delta^{k-1}}{2\pi^2} Z_\Lambda\{\tfrac{1}{2}(k+1)\}, \tag{15}$$

say, where $Z_\Lambda(s)$ is the Epstein zeta function

$$Z_\Lambda(s) = \sum_{\mathbf{m}}{}' \{Q(\mathbf{m})\}^{-s} \tag{16}$$

and has abscissa of convergence equal to $\frac{1}{2}k$.

In particular, it follows as in **(4)** and **(2)** from (10) and (14) that

$$\sigma^2 = O(\rho^{k-1}) \quad \text{and} \quad \sigma^2 = \Omega(\rho^{k-1})$$

as $\rho \to \infty$.

In the above work we have assumed that the lattice Λ is given and have associated a quadratic form $q(\mathbf{y})$ with it. It is clear that we could have adopted the reverse procedure and started from a positive definite quadratic form $q(\mathbf{y})$ and then associated a lattice with it, this lattice being unique apart from an arbitrary rotation.

4. In this section we obtain an alternative expression in finite closed form for the variance $\sigma^2(\rho, \Lambda)$ which is more suitable for calculation when ρ is small.

Let Π be any fundamental cell of the lattice Λ and let Π_0 be the parallelepiped obtained by translating Π so that its centre is at the origin. Let Π_N denote the parallelepiped obtained from Π_0 by magnification in the ratio $2N+1:1$ in all directions, where N is a large integer. The content of Π_N is $(2N+1)^k\Delta$ and it contains $(2N+1)^k$ points of Λ. Round each point of Λ we describe a hypersphere of fixed radius ρ. If ρ is sufficiently large, these hyperspheres will overlap. The region common to two spheres of radius ρ whose centres are at a positive distance t apart we call the *lune* defined by the spheres and denote it by $\mathscr{L}(\rho, t)$. The content of $\mathscr{L}(\rho, t)$ is

$$V(\rho, t) = 2 \int_{\frac{1}{2}t}^{\rho} K_{k-1}(\rho^2 - x^2)^{\frac{1}{2}(k-1)}\, dx = 2K_{k-1}\rho^k \int_0^{\sec^{-1}(2\rho/t)} \sin^k\theta\, d\theta. \tag{17}$$

Let P be a point of Π_N at which n spheres overlap. We assign to each of these n spheres a density $\frac{1}{2}(n-1)$ at P, so that the total density at P is $\frac{1}{2}n(n-1)$. But $\frac{1}{2}n(n-1)$ is the number of lunes which contain P as a point. Hence the total mass $\mathscr{M}$ contained in Π_N is equal to the sum of

the contents of all lunes or parts of lunes contained in Π_N. From the fact that the total density at P is $\frac{1}{2}n(n-1)$ we deduce that

$$\mathscr{M} = (2N+1)^k \Delta \sum_{n=0}^{\infty} \tfrac{1}{2}n(n-1)p_n(N), \tag{18}$$

where $p_n(N)$ denotes the probability that any point P is contained in exactly n spheres. The series on the right is, of course, a terminating series. As $N \to \infty$, $p_n(N) \to p_n$, the fraction of Π_N common to n spheres.

We now calculate $\mathscr{M}$ in a different way by estimating the contents of all the lunes. For fixed ρ there is only a finite number of different sizes of lune $\mathscr{L}(\rho, t)$, namely those for which $0 < t = \{q(\mathbf{m})\}^{\frac{1}{2}} \leqslant 2\rho$, where $\mathbf{m} \in \Lambda_0$. For $t \geqslant 0$, let $P(t)$ denote the number of $\mathbf{m} \in \Lambda_0$ for which $q(\mathbf{m}) = t^2$. Then each sphere contains $P(t)$ lunes $\mathscr{L}(\rho, t)$ with $0 < t \leqslant 2\rho$ and hence the total content of all the lunes contained in any one sphere is

$$\sum_{\substack{\mathbf{m} \in \Lambda_0 \\ 0 < \sqrt{q(\mathbf{m})} = t \leqslant 2\rho}} P(t) V(\rho, \sqrt{q(\mathbf{m})}) = \sum_{\mathbf{m}}{}' V(\rho, \sqrt{q(\mathbf{m})})$$

in the previous notation, since $V(\rho, t) = 0$ for $t > 2\rho$. As there are $(2N+1)^k$ spheres centred at points of Π_N and since each lune is defined by two spheres, it follows that†

$$\lim_{N \to \infty} \frac{\mathscr{M}}{(2N+1)^k} = \tfrac{1}{2} \sum_{\mathbf{m}}{}' V(\rho, \sqrt{q(\mathbf{m})}), \tag{19}$$

and therefore, from (18) and (19), we obtain the formula

$$\sum_{n=1}^{\infty} n(n-1)p_n = \frac{1}{\Delta} \sum_{\mathbf{m}}{}' V(\rho, \sqrt{q(\mathbf{m})}). \tag{20}$$

In a similar way, if we assign unit density to each sphere and let $\mathscr{M}'$ be the total mass of all the spheres or parts of spheres contained in Π_N, we find that

$$K_k \rho^k = \lim_{N \to \infty} \frac{\mathscr{M}'}{(2N+1)^k} = \Delta \sum_{n=0}^{\infty} n p_n,$$

so that we have the second formula

$$\sum_{n=0}^{\infty} n p_n = \frac{K_k \rho^k}{\Delta}. \tag{21}$$

Since $V(\rho, 0) = K_k \rho^k$, we see from (20) and (21) that

$$\sum_{n=0}^{\infty} n^2 p_n = \frac{1}{\Delta} \sum_{\mathbf{m} \in \Lambda_0} V(\rho, \sqrt{q(\mathbf{m})}). \tag{22}$$

† We use here the fact that the number of lunes overlapping the surface of Π_N is $O(N^{k-1})$ as $N \to \infty$.

We are now ready to estimate the variance. We have, by (21) and (22),

$$\begin{aligned}\sigma^2(\rho,\Lambda) &= \sum_{n=0}^{\infty} n^2 p_n - \{a(\mathbf{o})\}^2 \\ &= \frac{1}{\Delta}\sum_{\mathbf{m}\in\Lambda_0} V(\rho,\sqrt{q(\mathbf{m})}) - (K_k\rho^k/\Delta)^2 \\ &= \frac{K_k\rho^k}{\Delta}\left(1-\frac{K_k\rho^k}{\Delta}\right)+\frac{1}{\Delta}\sum_{\mathbf{m}}{}' V(\rho,\sqrt{q(\mathbf{m})}). \end{aligned} \tag{23}$$

Since the series on the right terminates when $q(\mathbf{m}) > 4\rho^2$, we have obtained a finite closed expression for $\sigma^2(\rho,\Lambda)$.

If we compare (10) and (23), we obtain the new identity

$$\begin{aligned}&\frac{K_k\rho^k}{\Delta}\left(1-\frac{K_k\rho^k}{\Delta}\right)+\frac{1}{\Delta}\sum_{\mathbf{m}}{}' V(\rho,\sqrt{q(\mathbf{m})}) \\ &\qquad = \Delta^{k-2}\rho^k \sum_{\mathbf{m}}{}' \left\{J_{\frac{1}{2}k}\left(\frac{2\pi\rho}{\Delta}\sqrt{Q(\mathbf{m})}\right)\right\}^2 \Big/ \{Q(\mathbf{m})\}^{\frac{1}{2}k}. \end{aligned} \tag{24}$$

5. In this section we consider in more detail the particular case when $k = 2$ and Λ is a plane lattice of determinant Δ. We then have, by (9), (10), (11), (12), and (14),

$$\mathscr{E}(l) = \pi\rho^2/\Delta, \tag{25}$$

$$\left.\begin{aligned}\sigma^2(\rho,\Lambda) &= \rho^2 \sum_{\mathbf{m}}{}' \left\{J_1\left(\frac{2\pi\rho}{\Delta}\sqrt{Q(\mathbf{m})}\right)\right\}^2 \Big/ Q(\mathbf{m}) \\ &= \rho A(\rho) + O(1) \end{aligned}\right\}, \tag{26}$$

where

$$A(\rho) = \frac{\Delta}{\pi^2}\sum_{\mathbf{m}}{}' \cos^2\left[\frac{2\pi\rho}{\Delta}\sqrt{Q(\mathbf{m})} - \tfrac{3}{4}\pi\right] \Big/ \{Q(\mathbf{m})\}^{\frac{3}{2}} \tag{27}$$

and

$$\bar{A} = \frac{\Delta}{2\pi^2}\sum_{\mathbf{m}}{}' \{Q(\mathbf{m})\}^{-\frac{3}{2}} = \frac{\Delta}{2\pi^2} Z_\Lambda(\tfrac{3}{2}). \tag{28}$$

Also, by (23),

$$\sigma^2 = \frac{\pi\rho^2}{\Delta}\left(1-\frac{\pi\rho^2}{\Delta}\right)+\frac{1}{\Delta}\sum_{\mathbf{m}}{}' V(\rho,\sqrt{q(\mathbf{m})}) \tag{29}$$

and

$$V(\rho,t) = \rho^2(2\phi - \sin 2\phi) \quad (t \leqslant 2\rho), \tag{30}$$

where $\phi = \sec^{-1}(2\rho/t)$.

Although the variance is a highly oscillatory function, it is natural to expect it to be, in some sense, least when Λ is the hexagonal lattice; more precisely, if we consider lattices of fixed determinant Δ, which we may without loss of generality assume to be $\frac{1}{2}\sqrt{3}$, it is to be expected

that $\bar{A}$, and therefore $Z_\Lambda(\frac{3}{2})$, will be least when Λ is the lattice defined by $\mathbf{a} = \mathbf{S}\mathbf{a}_1$, where

$$\mathbf{a}_1 = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{2}\sqrt{3} & 0 \end{pmatrix} \tag{31}$$

and $\mathbf{S}$ is any orthogonal matrix. That this is true has been shown by one of us (**6**), but no simple proof has been found.

6. We now consider the particular case when Λ is the hexagonal lattice defined by $\mathbf{x} = \mathbf{a}_1\mathbf{y}$, where $\mathbf{a}_1$ is given by (31). In this lattice the closest distance between points of the lattice is unity, and the associated forms are

$$q(\mathbf{y}) = y_1^2 - y_1y_2 + y_2^2, \qquad Q(\mathbf{y}) = y_1^2 + y_1y_2 + y_2^2. \tag{32}$$

For integral y_1, y_2 it is clear that $Q(\mathbf{y})$ assumes only integral values. We write $R(n)$ for the number of solutions of the Diophantine equation $Q(\mathbf{y}) = n$. It is known† that

$$R(n) = 6\sum_{d|n} \chi_1(d),$$

where $\chi_1(d)$ is the Dirichlet non-principal character associated with the modulus 3; i.e. $\chi_1(n) = \frac{2}{3}\sqrt{3}\sin\frac{2}{3}n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$. The associated Dirichlet series is

$$L(s, \chi_1) = \sum_{n=1}^{\infty} \chi_1(n)n^{-s}$$

and is convergent for $\mathrm{re}\, s > 0$ and absolutely convergent for $\mathrm{re}\, s > 1$. Then $Z_\Lambda(s)$ (see (16)) is the zeta function associated with the field generated by $\sqrt{3}$, and

$$Z_\Lambda(s) = 6\zeta(s)L(s, \chi_1), \tag{33}$$

where $\zeta(s)$ is the ordinary Riemann zeta function.

We now have, from (25), (26), (27), (28), and (29),

$$\mathscr{E}(l) = \frac{2\pi}{\sqrt{3}}\rho^2, \tag{34}$$

$$\left.\begin{aligned} \mathrm{var}(l) = \sigma^2 = \rho^2 \sum_{n=1}^{\infty} \frac{R(n)}{n}\{J_1(4\pi\rho\sqrt{(\tfrac{1}{3}n)})\}^2 \\ = \frac{2\pi\rho^2}{\sqrt{3}}\left(1 - \frac{2\pi\rho^2}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}} \sum_{0<n\leqslant 4\rho^2} R(n)V(\rho, \sqrt{n}) \\ = \rho A(\rho) + O(1) \end{aligned}\right\}, \tag{35}$$

where

$$A(\rho) = \frac{\sqrt{3}}{2\pi^2} \sum_{n=1}^{\infty} \frac{R(n)}{n^{\frac{3}{2}}} \cos^2[4\pi\rho\sqrt{(\tfrac{1}{3}n)} - \tfrac{3}{4}\pi], \tag{36}$$

and

$$\bar{A} = \frac{3\sqrt{3}}{2\pi^2}\zeta(\tfrac{3}{2})L(\tfrac{3}{2}, \chi_1). \tag{37}$$

† See, for example, Hecke (**1**), Kap. VII or Landau (**5**), Satz 204.

From Gram's table of the ζ-function we find that $\zeta(\frac{3}{2}) = 2{\cdot}612375$ and a computation analogous to that described in the Appendix to (**4**) shows that $L(\frac{3}{2}, \chi_1) = 0{\cdot}703968$. Thus

$$\bar{A} = 0{\cdot}48411. \tag{38}$$

The function $A(\rho)$ of (36) fluctuates about this average, and it is clear that

$$0 < A(\rho) < 2\bar{A} = 0{\cdot}96822. \tag{39}$$

It would be a difficult problem to evaluate

$$A_L = \underline{\lim_{\rho\to\infty}} A(\rho) \quad \text{and} \quad A_U = \overline{\lim_{\rho\to\infty}} A(\rho) \tag{40}$$

exactly. However, by using the fact that the square roots of the square-free numbers are linearly independent, in conjunction with Kronecker's theorem,† we can show that

$$A_L \leqslant B_L = \frac{\sqrt{3}}{2\pi^2}(S_3 + \tfrac{1}{2}S_2), \qquad A_U \geqslant B_U = \frac{\sqrt{3}}{2\pi^2}(S_1 + \tfrac{1}{2}S_2). \tag{41}$$

Here S_1, S_2, and S_3 are defined as follows. In the infinite series

$$S = \sum_{n=1}^{\infty} \frac{R(n)}{n^{\frac{3}{2}}}$$

write $n = qm^2$, where q is square-free. Then

$$S = \sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{\substack{q=1 \\ q \text{ square-free}}}^{\infty} \frac{R(m^2q)}{q^{\frac{3}{2}}},$$

and we denote by S_1 and S_3 the parts of this double series for which $m \equiv 1$ or $3 \pmod 4$, respectively; S_2 is the part for which m is even and is easily seen to be $\frac{1}{8}S$. Clearly

$$B_U + B_L = 2\bar{A} = \frac{3\sqrt{3}}{\pi^2}\zeta(\tfrac{3}{2})L(\tfrac{3}{2}, \chi_1).$$

$B_U - B_L$ can also be expressed as a rather complicated infinite Euler product. By direct calculation it can be shown that

$$0{\cdot}093 < B_L < 0{\cdot}115, \qquad 0{\cdot}854 < B_U < 0{\cdot}875. \tag{42}$$

7. Before discussing the relationship of the results of § 6 to the experimental circle-throwings carried out by Wright (**8**) it will be useful to put the variance in a form independent of the scale of the lattice. If we write

$$x = \sqrt{\mathscr{E}(l)} \quad \text{and} \quad y = \frac{\operatorname{var}(l)}{\sqrt{\mathscr{E}(l)}}, \tag{43}$$

† See, for example, Ingham (**3**) 134. We choose ρ to make $4\pi\rho\sqrt{(\frac{1}{4}n)} - \frac{3}{4}\pi$ approximately equal to an odd or even multiple of $\frac{1}{2}\pi$ for square-free n when considering the lower or upper limits respectively.

then estimates of these two quantities will be observable even when the scale of the lattice is unknown, and so the theoretical relation between them may be of importance in interpreting the experimental cutaneous-sensation curves not considered here. Now

$$x = \frac{\sqrt{(2\pi)}}{3^{\frac{1}{4}}}\rho = 1{\cdot}9046\rho, \tag{44}$$

and

$$y = \frac{3^{\frac{1}{4}}}{\sqrt{(2\pi)}}A(\rho) + O\left(\frac{1}{\rho}\right), \tag{45}$$

by (35). Thus† the x-average of y is given by

$$Y = \frac{3^{\frac{1}{4}}}{\sqrt{(2\pi)}}\bar{A} = 0{\cdot}5250\bar{A} \tag{46}$$

(this is also equal to the average of y with respect to x^2). The analogous result for the *square* lattice can be derived from the formulae established in (**4**); in fact we find that for this lattice, $Y = a^2/\sqrt{\pi}$, where $a = 0{\cdot}676497$. The numerical values are given in the following table.

TABLE 1

Lattice	*Theoretical value for Y*
Hexagonal	0·254 17
Square	0·258 20

Wright has made 23,000 random throwings of circles of various sizes on to a hexagonal lattice and has recorded the numbers of lattice-points caught at each throw. His results are shown (as black spots) on an (x, y)-diagram in Fig. 1.

When $\rho < \frac{1}{2}$, the circle can contain one lattice-point only or none, and one easily finds from this that

$$y = x(1-x^2) \tag{47}$$

when $0 \leqslant x \leqslant x_1 = \sqrt{(\frac{1}{2}\pi)}/3^{\frac{1}{4}} = 0{\cdot}95231$, as has been pointed out to us by Wright. The expression on the right of (47) corresponds, of course, to the first term on the right-hand side of the second formula in (35). From it we deduce the general formula

$$y = x\left\{1-x^2+\frac{1}{\pi}\sum_{0<n\leqslant\frac{2\sqrt{3}}{\pi}x^2} R(n)(2\phi_n-\sin 2\phi_n)\right\}, \tag{48}$$

where

$$\phi_n = \sec^{-1}\left\{x\sqrt{\left(\frac{2\sqrt{3}}{\pi n}\right)}\right\}.$$

† There is no difficulty at $x = 0$, despite the term $O(1/\rho)$, because (as we shall shortly see) y is a bounded continuous function of x for $x \geqslant 0$.

From this formula the part of the (x, y)-curve in the range

$$x \leqslant (19\pi/2\sqrt{3})^{\frac{1}{2}} = 4{\cdot}151$$

has been drawn and is shown by the continuous line in Fig. 1. It will be seen from the figure that Wright's experimental determinations were remarkably accurate.

A comparison of the experimental and theoretical data represented by Fig. 1 with our conclusion about the behaviour of the almost-periodic

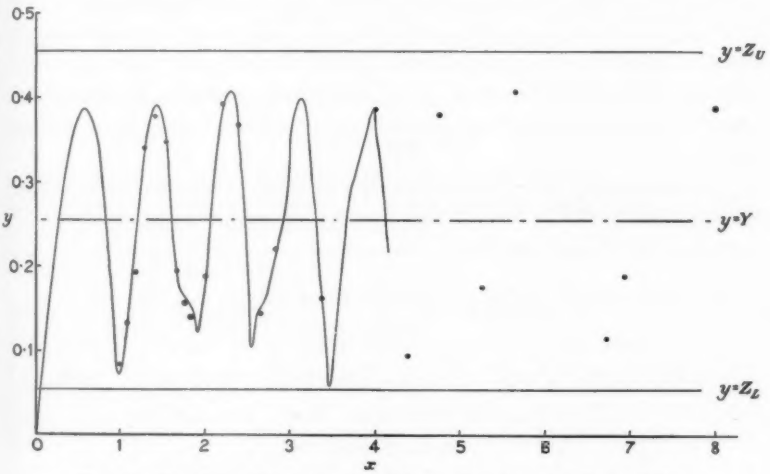

FIG. 1. Number of hexagonal lattice points in a random circle.
Ordinate y = variance/√(mean), abscissa x = √(mean).
Experiments by G. H. Wright •; Theory ———.

function $A(\rho)$ is not quite fair because, as equations (45) and (46) make evident, the behaviour of $y(x)$ is asymptotically equivalent to that of $A(\rho)$ only when x is large. However, we shall then have

$$Y_U \equiv \limsup_{x\to\infty} y(x) = \frac{3^{\frac{1}{2}}}{\sqrt{(2\pi)}} A_U \geqslant \frac{3^{\frac{1}{2}}}{\sqrt{(2\pi)}} B_U \equiv Z_U,$$

and

$$Y_L \equiv \liminf_{x\to\infty} y(x) = \frac{3^{\frac{1}{2}}}{\sqrt{(2\pi)}} A_L \leqslant \frac{3^{\frac{1}{2}}}{\sqrt{(2\pi)}} B_L \equiv Z_L;$$

and it is of interest to compare the fluctuations of $y(x)$ with Y and with Z_L and Z_U when $0 \leqslant x \leqslant 4{\cdot}151$. The value of Y and the (approximate) values of Z_L and Z_U have accordingly been indicated in Fig. 1.

8. We think it of interest to set down the particular forms which the identity (24) takes for the hexagonal and square lattices respectively. The formulae are

$$\sum_{n=1}^{\infty} \frac{R(n)}{n} J_1^2(2\pi\sqrt{(\tfrac{1}{3}nx)}) = \frac{2\pi}{\sqrt{3}}\left(1-\frac{\pi x}{2\sqrt{3}}\right)+\frac{2}{\sqrt{3}}\sum_{0<n\leqslant x} R(n)(2\theta_n-\sin 2\theta_n), \tag{49}$$

and

$$\sum_{n=1}^{\infty} \frac{r(n)}{n} J_1^2(\pi\sqrt{(nx)}) = \pi(1-\tfrac{1}{4}\pi x) + \sum_{0<n\leqslant x} r(n)(2\theta_n-\sin 2\theta_n). \tag{50}$$

Here

$$\theta_n = \sec^{-1}\{\sqrt{(x/n)}\} \tag{51}$$

and $r(n)$ denotes the number of representations of n as a sum of two squares; the variable x has no connexion with the quantity x defined in § 7.

In conclusion, we remark that the identity (50) can be obtained directly, without the use of the concept of variance, as follows. By equation (9) of Ingham (3)

$$\int_a^b \chi(t)\mathrm{P}(t)\,dt = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n}\int_a^b \chi(t)f(nt)\,dt, \tag{52}$$

where $b > a \geqslant 0$, $\chi(t)$ is a function possessing a continuous derivative for $a \leqslant t \leqslant b$, $f(v) = \pi v^{\frac{1}{2}}J_1(2\pi v^{\frac{1}{2}})$ and

$$\mathrm{P}(t) = \sum_{n\leqslant t} r(n) - \pi t.$$

We take $a = \epsilon > 0$, $b = x$, $\chi(t) = \{(x-t)/t\}^{\frac{1}{2}}$ and obtain, from (52),

$$\int_\epsilon^x (x-t)^{\frac{1}{2}}t^{-\frac{1}{2}}\Big\{\sum_{n\leqslant t} r(n)-\pi t\Big\}\,dt = \sum_{n=1}^{\infty}\frac{r(n)}{n^{\frac{1}{2}}}\int_\epsilon^x (x-t)^{\frac{1}{2}}J_1\{2\pi\sqrt{(nt)}\}\,dt. \tag{53}$$

By using the inequality

$$|J_\nu(z)| \leqslant C\min(z^\nu, z^{-\frac{1}{2}}) \quad (z>0),$$

where C is a constant, it is possible to show that the error made in replacing ϵ by 0 on each side of (53) is $O(\epsilon^{\frac{1}{2}})$ as $\epsilon \to +0$, and hence we obtain

$$\pi(1-\tfrac{1}{4}\pi x) + \sum_{0<n\leqslant x} r(n)(2\theta_n-\sin 2\theta_n) = \int_0^x (x-t)^{\frac{1}{2}}t^{-\frac{1}{2}}\Big\{\sum_{n\leqslant t} r(n)-\pi t\Big\}\,dt$$

$$= \sum_{n=1}^{\infty}\frac{r(n)}{n^{\frac{1}{2}}}\int_0^x (x-t)^{\frac{1}{2}}J_1\big(2\pi\sqrt{(nt)}\big)\,dt.$$

If we make the substitution $t = x\operatorname{cosec}^2 \frac{1}{2}\phi$ in this last integral and use the formula [see equation (16) of § 11.41 of (7)]

$$\frac{J_1^2(\pi\sqrt{(nx)})}{\sqrt{(nx)}} = \int_0^\pi \frac{\sin^2\phi}{\sin\frac{1}{2}\phi} J_1\{2\pi\sqrt{(nx)}\sin\tfrac{1}{2}\phi\}\, d\phi,$$

we obtain (50). The identity (49) can be derived in the same way, and doubtless it is possible to deduce the more general result (24) by a similar argument.

We should like to thank Mr. G. H. Wright and Dr. H. E. Daniels for suggesting that we should consider this problem. We are also very grateful to Mr. Wright for making available to us the detailed records of his circle-throwing experiments.

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ON A GENERALIZED DERIVATIVE

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[Received 3 January 1953]

1. SEVERAL well-known generalizations of the nth derivative of a function $f(x)$ are obtained by a single limiting process applied to one of the nth differences of $f(x)$. For example, the symmetric derivative $D^n f(x)$ is defined in this manner: the nth symmetric difference is

$$\Delta^n f(x,h) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x-rh+\tfrac{1}{2}nh),$$

and

$$D^n f(x) = \lim_{h\to 0} h^{-n}\Delta^n f(x,h) \tag{1}$$

if this limit exists. Any linear operator which, like Δ^n, annihilates polynomials of degree less than n may be called an 'nth difference', and by using such differences instead of Δ^n we obtain a large class of generalized nth derivatives. It is the purpose of this note to discuss one such derivative and to show that it is, in a sense, the weakest member of the class.

2. The general class of differences may be defined as follows. Let $\mathscr{B}_n$ be the set of functions of a real variable t such that $\alpha(t) \in \mathscr{B}_n$ if and only if $\alpha(t)$ is of bounded variation in the interval $-1 \leqslant t \leqslant 1$ and

$$\left.\begin{aligned} \int_{-1}^{1} t^r \, d\alpha(t) = 0 \quad (r = 1, 2, \ldots, n-1) \\ \int_{-1}^{1} d\alpha(t) = 0, \qquad \int_{-1}^{1} t^n \, d\alpha(t) = n! \end{aligned}\right\}. \tag{2}$$

If $f(x)$ is continuous in an interval $(c-\delta, c+\delta)$ and if $\alpha(t) \in \mathscr{B}_n$, then

$$\Delta_\alpha f(c,h) \equiv \int_{-1}^{1} f(c+ht) \, d\alpha(t) \tag{3}$$

exists for $|h| < \delta$; and, since $\Delta_\alpha f(c,h) = 0$ whenever $f(x)$ is a polynomial of degree less than n, it may be called an nth difference of $f(x)$. In particular, the symmetric difference $\Delta^n f(c,h) = \Delta_\beta(c, \frac{1}{2}nh)$ if $\beta(t)$ is a suitably-chosen step-function.

By analogy with (1) we may define an 'nth derivative' corresponding to any such difference.

Quart. J. Math. Oxford (2), 4 (1953), 190–7.

DEFINITION A. *Let $\alpha(t) \in \mathscr{B}_n$, and let $f(x)$ be such that*

$$\Delta_\alpha f(c,h) \equiv \int_{-1}^{1} f(c+ht)\, d\alpha(t)$$

exists for h sufficiently small: then

$$D_\alpha f(c) = \lim_{h\to 0} h^{-n}\Delta_\alpha f(c,h)$$

if this limit exists.

It is sufficient for the existence of $\Delta_\alpha f(c,h)$ that $f(x)$ be Borel-measurable and bounded in the neighbourhood of $x = c$, but we shall here suppose, in general, that $f(x)$ is continuous.

It is easy to show that, if the ordinary derivative $f^{(n)}(c)$ exists, then $D_\alpha f(c)$ exists (and is equal to $f^{(n)}(c)$) for every $\alpha(t) \in \mathscr{B}_n$. In fact, a less strict condition is sufficient: namely, the existence of the nth 'Peano derivative' (here denoted by $T_n f(c)$) which is defined as follows:

DEFINITION B. *Let $f(x)$ be defined in an interval $(c-\delta, c+\delta)$ and let*

$$f(c+h) = a_0+a_1 h+\ldots+a_n h^n/n!+\epsilon(h)h^n \tag{4}$$

for $|h| < \delta$, where $a_0, a_1, \ldots, a_n$ are independent of h and $\epsilon(h) \to 0$ as $h \to 0$: then $a_n = T_n f(c)$ is called the n-th Peano derivative of $f(x)$ at $x = c$.

If $T_n f(c)$ exists, then we may use (4) to expand $f(c+ht)$ in (3) and deduce from (2) that $h^{-n}\Delta_\alpha f(c,h) \to T_n f(c)$ as $h \to 0$; that is, the existence of $T_n f(c)$ and the continuity of $f(x)$ together imply the existence of $D_\alpha f(c)$ for every $\alpha(t) \in \mathscr{B}_n$. Hence, if we write $\mathscr{T}_n(c)$, $\mathscr{D}_\alpha(c)$ for the classes of continuous functions $f(x)$ such that $T_n f(c)$, $D_\alpha(c)$ respectively exist, then $\mathscr{T}_n(c) \subset \mathscr{D}_\alpha(c)$ whenever $\alpha(t) \in \mathscr{B}_n$. The theorem stated in § 4 below shows that *there is an $\omega(t) \in \mathscr{B}_n$ such that the classes $\mathscr{T}_n(c)$ and $\mathscr{D}_\omega(c)$ are identical*, and therefore $\mathscr{D}_\omega(c) \subset \mathscr{D}_\alpha(c)$ for every other $\alpha(t) \in \mathscr{B}_n$. Before proving the theorem I discuss in § 3 two of the conditions which $\omega(t)$ must necessarily satisfy if the operator D_ω is to be equivalent to T_n.

3. Necessary conditions

We may first simplify the discussion by supposing that $c = 0$, and for brevity writing $\Delta_\alpha f(h)$ for $\Delta_\alpha f(0,h)$. Moreover, if $D_\alpha f(0)$ exists and we write

$$\phi(x) = f(x)-x^n D_\alpha f(0)/n!,$$

then $D_\alpha \phi(0)$ exists and is zero. It is therefore sufficient to find $\omega(t)$ such that $T_n \phi(0)$ exists and is zero whenever $D_\omega \phi(0) = 0$: that is, such that, if $\phi(x)$ is continuous and

$$\Delta_\omega \phi(h) \equiv \int_{-1}^{1} \phi(ht)\, d\omega(t) = o(h^n) \tag{5}$$

as $h \to 0$, then

$$\phi(h) = p_0 + p_1 h + \ldots + p_{n-1} h^{n-1} + o(h^n), \tag{6}$$

where $p_0, p_1, \ldots, p_{n-1}$ are independent of h.

The problem is further simplified if we separate the even and odd parts of $\phi(x)$ and $\omega(t)$: we may then suppose that $h > 0$. If $F(y)$ is any given function, let

$$F_1(y) = \tfrac{1}{2}\{F(y) + F(-y)\}, \qquad F_2(y) = \tfrac{1}{2}\{F(y) - F(-y)\}.$$

Then, if (5) is true,

$$\int_0^1 \phi_1(ht)\, d\omega_2(t) = \tfrac{1}{4} \int_{-1}^1 \{\phi(ht) + \phi(-ht)\}\, d\omega(t)$$

$$= \tfrac{1}{4}\Delta_\omega \phi(h) + \tfrac{1}{4}\Delta_\omega \phi(-h) = o(h^n); \tag{7}$$

and similarly

$$\int_0^1 \phi_2(ht)\, d\omega_1(t) = \tfrac{1}{4}\Delta_\omega \phi(h) - \tfrac{1}{4}\Delta_\omega \phi(-h) = o(h^n). \tag{8}$$

Also, if (7) and (8) are simultaneously true as $h \to +0$, then (5) is true as $h \to 0$.

There are two simple conditions which $\omega_1(t)$ and $\omega_2(t)$ must necessarily satisfy if (6) is to be a consequence of (5), or equivalently of (7) and (8):

CONDITION I. *If* $\qquad \Omega_i(z) = \int_0^1 t^z\, d\omega_i(t),$

then $\Omega_1(z)$ (*which is regular in the half-plane* $\operatorname{re} z > 0$) *must have no zeros in the strip* $0 < \operatorname{re} z \leqslant n$ *other than simple zeros at* $z = 1, 3, 5, \ldots$; *similarly,* $\Omega_2(z)$ *must have no zeros in this strip other than simple zeros at* $z = 2, 4, 6, \ldots$. (*Also, of course, if* $\omega(t) \in \mathscr{B}_n$, $\Omega_2(0) = 0$.)

For example, let $\Omega_1(\xi + i\eta) = 0$, where $0 < \xi \leqslant n$ and, if $\eta = 0$, ξ is not an odd integer. Let $\phi_1(x) = 0$ and $\phi_2(x) = \operatorname{re}(x^{\xi+i\eta}) = x^\xi \cos(\eta \log x)$, when $x > 0$. Then $\Delta_\omega \phi(h) = 0$, but $T_n \phi(0)$ does not exist. Again, let $\Omega_1(z)$ have a double zero at $z = 2r-1$, where $0 < 2r-1 \leqslant n$: then

$$\int_0^1 t^{2r-1}\, d\omega_1(t) = 0, \qquad \int_0^1 t^{2r-1} \log t\, d\omega_1(t) = 0;$$

and so $\qquad \phi_2(x) = x^{2r-1} \log x \quad (x > 0), \qquad \phi_1(x) = 0$

again gives $\Delta_\omega \phi(h) = 0$ and $T_n \phi(0)$ does not exist. Similar examples show the necessity of the conditions on $\Omega_2(z)$.

CONDITION II. *Each of $\omega_1(t)$, $\omega_2(t)$ must have at least one discontinuity in the half-open interval $0 < t \leqslant 1$.*

Suppose that $\alpha(t)$ is a function which is continuous in $0 < t \leqslant 1$ and of bounded variation in $0 \leqslant t \leqslant 1$: then, *given any positive numbers n, ϵ, there exists a continuous function $\psi(x)$ such that, as $h \to +0$,*

$$\int_0^1 \psi(ht)\, d\alpha(t) = o(h^n), \tag{9}$$

and $\psi(h) \to 0$, but $h^{-\epsilon}\psi(h) \not\to 0$. Let $\beta(u)$ be the variation of $\alpha(t)$ in the interval $0 \leqslant t \leqslant u$; then $\beta(u)$ is monotone, bounded, and continuous in $0 < u \leqslant 1$, and so is uniformly continuous in this interval. Therefore for each integer r there exists δ_r (where $0 < 4\delta_r < 1$) such that

$$0 \leqslant \beta(u') - \beta(u) \leqslant 2^{-nr}$$

whenever $0 < u \leqslant u' \leqslant 1$ and $u' - u \leqslant 4\delta_r$. For $r = 1, 2, 3, \ldots$ let

$$\psi(2^{-r}) = 2^{-\epsilon r}, \qquad \psi\{2^{-r}(1 \pm \delta_r)\} = 0,$$

let $\psi(0) = \psi(1) = 0$, and let $\psi(x)$ be linear in each of the remaining intervals of $(0, 1)$. Then $\psi(x)$ is continuous and, as $h \to +0$, $\psi(h) \to 0$ but $h^{-\epsilon}\psi(h)$ oscillates between 0 and 1. Also, if $0 < h < 1$ and we define $\beta(t) = \beta(1)$ for $t > 1$,

$$0 \leqslant \int_0^1 \psi(ht)\, d\beta(t) = \sum_{r=1}^{\infty} \int_{2^{-r}(1-\delta_r)h^{-1}}^{2^{-r}(1+\delta_r)h^{-1}} \psi(ht)\, d\beta(t)$$

$$\leqslant \sum_{r=1}^{\infty} 2^{-\epsilon r}\left[\beta\{2^{-r}(1+\delta_r)h^{-1}\} - \beta\{2^{-r}(1-\delta_r)h^{-1}\}\right].$$

In this last sum all terms for which $2^r h < \frac{1}{2}$ are zero; by the definition of δ_r each remaining term $\leqslant 2^{-(n+\epsilon)r}$. Therefore, if R is the least integer such that $2^R h \geqslant \frac{1}{2}$, we have

$$0 \leqslant \int_0^1 \psi(ht)\, d\beta(t) \leqslant \sum_{r=R}^{\infty} 2^{-(n+\epsilon)r} < 2^{-(n+\epsilon)R+1} = o(h^n).$$

A fortiori, since $\psi(ht) \geqslant 0$ and $\beta(t)$ is the variation of $\alpha(t)$, (9) is true.

Condition II is now easily deduced. If $\omega_1(t)$ is continuous in $0 < t \leqslant 1$, let $\phi_2(x)$ be equal (for $x > 0$) to the function $\psi(x)$ defined above with $\alpha(t) = \omega_1(t)$ and some $\epsilon < 1$; and let $\phi_1(x) = 0$. Then

$$\int_{-1}^1 \phi(ht)\, d\omega(t) = 2\int_0^1 \phi_2(ht)\, d\omega_1(t) = o(h^n),$$

but $T_n\phi(0)$ does not exist. A similar counter-example shows that $\omega_2(t)$ must not be continuous.

4. There is no reason to suppose that the Conditions I and II are sufficient as well as necessary, but we shall see that a very simple $\omega(t)$ satisfying these conditions does in fact yield a D_ω which is equivalent to T_n. We take $\omega(t) = p(t)$ when $-1 \leqslant t < 1$, where $p(t)$ is a polynomial, and $\omega(1) = 1+p(1)$: then

$$\int_{-1}^{1} \phi(ht)\, d\omega(t) = \phi(h) + \int_{-1}^{1} \phi(ht) p'(t)\, dt.$$

It is easy to verify (as in § 5 below) that, if $p'(t)$ is the polynomial of lowest degree such that $\Omega_1(z)$ and $\Omega_2(z)$ have the required zeros, then

$$p'(t) = -\tfrac{1}{2}\{P'_n(t)+P'_{n-1}(t)\},$$

where $P_n(t)$ is the nth Legendre polynomial.

THEOREM. *If $\phi(x)$ is continuous for $-\delta < x < \delta$, and if, as $h \to 0$,*

$$g(h) \equiv \phi(h)-\tfrac{1}{2}\int_{-1}^{1} \phi(ht)\{P'_n(t)+P'_{n-1}(t)\}\, dt = o(h^n), \tag{10}$$

then there exist constants $p_0, p_1, \ldots, p_{n-1}$ such that

$$\phi(h) = p_0+p_1 h+\ldots+p_{n-1}h^{n-1}+o(h^n) \tag{11}$$

as $h \to 0$; that is, $T_n\phi(0)$ exists and is zero.

The proof of the theorem depends on the solution of a simple integral equation given (in a rather more general form than is required here) in Lemmas 1 and 2.

5. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be real constants such that

$$0 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_n, \qquad 0 \leqslant \beta_1 < \beta_2 < \ldots < \beta_n, \qquad \alpha_1+\beta_1 > 0;$$

let
$$R_{\alpha,\beta}(t) = R(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; t) = \sum_{i=1}^{n} b_i t^{\beta_i-1}, \tag{12}$$

where the coefficients b_i are such that

$$1 - \int_0^1 t^z R_{\alpha,\beta}(t)\, dt = \frac{(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)}{(z+\beta_1)(z+\beta_2)\ldots(z+\beta_n)}; \tag{13}$$

that is,
$$b_i = (-1)^{n-1} \prod_{r=1}^{n} (\alpha_r+\beta_i) \Big/ \prod_{r=1}^{i-1} (\beta_r-\beta_i) \prod_{r=i+1}^{n} (\beta_r-\beta_i). \tag{14}$$

From (13) we have, for $\lambda > 0$,

$$\int_0^1 t R_{\beta,\alpha}(t) R_{\alpha,\beta}(\lambda t)\, dt = R_{\alpha,\beta}(\lambda); \tag{15}$$

$$\int_0^{\lambda^{-1}} t R_{\beta,\alpha}(t) R_{\alpha,\beta}(\lambda t)\, dt = \lambda^{-2} R_{\beta,\alpha}(\lambda^{-1}). \tag{16}$$

LEMMA 1. *If $\phi(t)$ is Lebesgue-integrable over $0 \leqslant t \leqslant k$, if $\beta_1 \geqslant 1$, and*

$$g(h) = \phi(h) - \int_0^1 \phi(ht) R_{\alpha,\beta}(t)\, dt \quad (0 < h \leqslant k), \tag{17}$$

then, for $0 < h \leqslant k$, $0 < Xh \leqslant k$,

$$g(h) - \int_1^X g(hu) u^{-2} R_{\beta,\alpha}(u^{-1})\, du$$
$$= \phi(h) - h \int_0^{hX} \phi(v) \int_0^{(hX)^{-1}} w R_{\beta,\alpha}(wh) R_{\alpha,\beta}(vw)\, dw dv. \tag{18}$$

Proof.

$$\int_1^X u^{-2} R_{\beta,\alpha}(u^{-1}) \int_0^1 \phi(hut) R_{\alpha,\beta}(t)\, dt du$$
$$= \int_{X^{-1}}^1 \mu R_{\beta,\alpha}(\mu) \int_0^{\mu^{-1}} \phi(h\lambda) R_{\alpha,\beta}(\lambda\mu)\, d\lambda d\mu$$

(by substituting $u = \mu^{-1}$, $t = \lambda\mu$)

$$= \left\{ -\int_0^X \int_0^{X^{-1}} + \int_0^1 \int_0^1 + \int_1^X \int_0^{\lambda^{-1}} \right\} \phi(h\lambda) \mu R_{\beta,\alpha}(\mu) R_{\alpha,\beta}(\lambda\mu)\, d\mu d\lambda$$

(by changing the order of integration)

$$= -I_1 + I_2 + I_3, \text{ say.}$$

$$I_1 = \int_0^X \phi(h\lambda) \int_0^{X^{-1}} \mu R_{\beta,\alpha}(\mu) R_{\alpha,\beta}(\lambda\mu)\, d\mu d\lambda$$
$$= h \int_0^{hX} \phi(v) \int_0^{(hX)^{-1}} w R_{\beta,\alpha}(wh) R_{\alpha,\beta}(vw)\, dw dv,$$

by writing $\lambda = vh^{-1}$, $\mu = wh$.

By (15) we have

$$I_2 = \int_0^1 \phi(h\lambda) \int_0^1 \mu R_{\beta,\alpha}(\mu) R_{\alpha,\beta}(\lambda\mu)\, d\mu d\lambda$$
$$= \int_0^1 \phi(h\lambda) R_{\alpha,\beta}(\lambda)\, d\lambda$$
$$= \phi(h) - g(h).$$

Similarly, by (16),

$$I_3 = \int_1^X \phi(h\lambda) \lambda^{-2} R_{\beta,\alpha}(\lambda^{-1})\, d\lambda.$$

Combining these results, we have

$$\int_1^X g(hu)u^{-2}R_{\beta,\alpha}(u^{-1})\,du$$

$$= \int_1^X \phi(hu)u^{-2}R_{\beta,\alpha}(u^{-1})\,du + I_1 - I_2 - I_3$$

$$= I_1 - \phi(h) + g(h),$$

from which the required formula (18) follows immediately.

LEMMA 2. *If the conditions of Lemma* 1 *are satisfied and if*

$$\int_{\to 0}^1 g(u)u^{-\alpha_n - 1}\,du \tag{19}$$

is convergent, then there are constants $q_0, q_1, \ldots, q_n$ *such that, for* $0 < h \leqslant k$,

$$\phi(h) = \sum_{i=1}^n q_i h^{\alpha_i} + g(h) + \int_{\to 0}^1 g(hu)u^{-2}R_{\beta,\alpha}(u^{-1})\,du. \tag{20}$$

Proof. If we write $hX = \eta$ in (18), then, for $0 < h \leqslant k$, $0 < \eta \leqslant k$, we have

$$g(h) + \int_{\eta/h}^1 g(hu)u^{-2}R_{\beta,\alpha}(u^{-1})\,du \tag{21}$$

$$= \phi(h) - h\int_0^\eta \phi(v)\int_0^{\eta^{-1}} wR_{\beta,\alpha}(wh)R_{\alpha,\beta}(vw)\,dwdv$$

$$= \phi(h) - \sum_{i=1}^n c_i h^{\alpha_i}\int_0^\eta \phi(v)\int_0^{\eta^{-1}} w^{\alpha_i}R_{\alpha,\beta}(vw)\,dwdv$$

$$= \phi(h) - \sum_{i=1}^n q_i(\eta)h^{\alpha_i}, \tag{22}$$

where c_i is the coefficient of $h^{\alpha_i - 1}$ in $R_{\beta,\alpha}(h)$, and $q_i(\eta)$ is independent of h. If h is fixed and $\eta \to 0$, then, by (19), the integral in (21) tends to a limit. Therefore the polynomial in (22) also tends to a limit as $\eta \to 0$ for each fixed value of h. Hence each coefficient $q_i(\eta)$ tends to a limit, say q_i, independent of h; and so, if $0 < h \leqslant k$,

$$g(h) + \int_{\to 0}^1 g(hu)u^{-2}R_{\beta,\alpha}(u^{-1})\,du = \phi(h) - \sum_{i=1}^n q_i h^{\alpha_i},$$

which is the required result.

6. Proof of the Theorem

From (10) we deduce, using the notation of § 3 for the even and odd part of $\phi(h)$ and $g(h)$, that, if n is even,

$$\left.\begin{aligned} g_1(h) &= \phi_1(h) - \int_0^1 \phi_1(ht) P'_{n-1}(t)\, dt \\ g_2(h) &= \phi_2(h) - \int_0^1 \phi_2(ht) P'_n(t)\, dt \end{aligned}\right\}. \tag{23}$$

If n is odd, then n and $n-1$ are interchanged in these formulae.

Now, for $\operatorname{re} z > 0$,

$$1 - \int_0^1 t^z P'_{2r}(t)\, dt = z \int_0^1 t^{z-1} P_{2r}(t)\, dt,$$

which is zero for $z = 1, 3, \ldots, 2r-1$. Also integration of $t^z P'_{2r}(t)$ shows that the expression on the left is of the form

$$1 - \frac{k_1}{z+2} - \frac{k_2}{z+4} - \ldots - \frac{k_r}{z+2r},$$

and so must be $$\frac{(z-1)(z-3)\ldots(z-2r+1)}{(z+2)(z+4)\ldots(z+2r)}.$$

Therefore
$$P'_{2r}(t) = R(1, 3, \ldots, 2r-1; 2, 4, \ldots, 2r; t),$$
$$t^{-1} P'_{2r}(t) = R(2, 4, \ldots, 2r; 1, 3, \ldots, 2r-1; t).$$

Similarly
$$P'_{2r-1}(t) = R(0, 2, 4, \ldots, 2r-2; 1, 3, \ldots, 2r-1; t),$$
$$t^{-1} P'_{2r-1}(t) = R(1, 3, \ldots, 2r-1; 0, 2, 4, \ldots, 2r-2; t).$$

Using these forms for $R_{\alpha,\beta}(t)$ and $R_{\beta,\alpha}(t)$ in Lemma 2 we deduce from (23) that, if $n = 2r$ and $h > 0$,

$$\left.\begin{aligned} \phi_1(h) &= \sum_{i=0}^{r-1} p_{2i} h^{2i} + g_1(h) + \int_{\to 0}^1 g_1(hu) u^{-1} P'_{2r-1}(u^{-1})\, du \\ \phi_2(h) &= \sum_{i=0}^{r-1} p_{2i+1} h^{2i+1} + g_2(h) + \int_{\to 0}^1 g_2(hu) u^{-1} P'_{2r}(u^{-1})\, du \end{aligned}\right\}, \tag{24}$$

provided that
$$\int_{\to 0}^1 g_s(u) u^{-2r}\, du \quad (s = 1, 2)$$

is convergent. Since $g(u) = o(u^{2r})$ as $u \to 0$, the last condition is satisfied and also each integral in (24) is $o(h^{2r})$. By addition and subtraction of the equations (24) we deduce that, both for $h > 0$ and for $h < 0$,

$$\phi(h) = \sum_{j=0}^{2r-1} p_j h^j + o(h^{2r}),$$

which is the required result when $n = 2r$. A similar proof establishes the theorem for $n = 2r+1$.

UNIFORM DIMENSION AND THE PRODUCT THEOREM

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[Received 7 January 1953; in revised form 6 May 1953]

Introduction

THE inductive definition of dimension says that to show that a topological space X has dimension $\leqslant n$ we must produce a base of open sets $\{G\}$ such that each $\mathfrak{B}G$ has dimension $\leqslant n-1$: then to show that this is so, we must select a new base for each $\mathfrak{B}G$ whose sets have boundaries with dimension $\leqslant n-2$; and so on. ($\mathfrak{B}G$ denotes the boundary of G; $\mathfrak{B}_Y G$ the boundary of G in a subspace $Y \supset G$.) Suppose we insist that, having once chosen our base $\{G\}$, we will be lazy and use the base $\{G \cap \mathfrak{B}G'\}$ to display the $(n-1)$-dimensionality of $\mathfrak{B}G'$, and similarly at each of the $n+1$ stages: this may be expected to give a different value for the dimension, greater than or equal to the usual one. This I shall call the *uniform dimension* of X, and denote by $\operatorname{Dim} X$. (The formal definition is given in § 1.1, below.)

It follows from a result proved by Eilenberg and Otto [(**1**) Cor. 2], stated below as Theorem 2.7, that, for separable metric spaces, dimension and uniform dimension agree. In this paper an alternative proof of this result is given, using the embedding theorem instead of the decomposition theorem of Kuratowski on which Eilenberg and Otto's results are based; and a proof of the product theorem for uniform dimension is given that does not require the use of a sum theorem.

1. Definition and immediate results

1.1. We define first the dimension of a subset *over* a given base. Following the pattern of Hurewicz and Wallman [(**2**), Definition III.1], we say:

If X is a topological space, $\{G\}$ a collection of its open sets, and Y any subset (possibly equal to X), then $\operatorname{Dim} Y/\{G\} = -1$ *if and only if* $Y = \phi$; *and* $\operatorname{Dim} Y/\{G\} \leqslant n$ *if and only if* $\{Y \cap G\}$ *is a base for Y and, for each G,* $\operatorname{Dim} \mathfrak{B}_Y(Y \cap G)/\{G\} \leqslant n-1$.

Now we define $\operatorname{Dim} Y$ as *the least value of* $\operatorname{Dim} Y/\{G\}$ *for any collection*

$\{G\}$ for which it exists. Thus $\operatorname{Dim} X \leqslant n$ if and only if there is a base $\{G\}$ for X such that, for each G,

$$\operatorname{Dim} \mathfrak{B}G/\{G\} \leqslant n-1.$$

Despite the form of the definition, it is easily proved by induction that $\operatorname{Dim} Y$ is independent of the containing space X, and that $\operatorname{Dim} Y/\{G\}$ depends only on Y and the collection $\{Y \cap G\}$.

1.2. *The monotone property of dimension holds for uniform dimension: i.e. if $Y \subset X$ and $\operatorname{Dim} X/\{G\} = n$, then $\operatorname{Dim} Y/\{G\} \leqslant n$.*

The proof is by induction. The result is trivial if $n = -1$, i.e. $X = \phi$; suppose it true if n be replaced by $n-1$. For each G,

$$\operatorname{Dim} \mathfrak{B}G/\{G\} \leqslant n-1;$$

but clearly $\mathfrak{B}_Y(Y \cap G) \subset \mathfrak{B}G$: so the induction hypothesis gives

$$\operatorname{Dim} \mathfrak{B}_Y(Y \cap G)/\{G\} \leqslant n-1,$$

and the result follows.

We deduce the corollary: *if $\operatorname{Dim} X = n$, and $Y \subset X$, then $\operatorname{Dim} Y \leqslant n$.*

1.3. A second kind of monotone theorem can also be proved by induction; viz. *$\operatorname{Dim} X/\{G\}$ is not increased if $\{G\}$ is replaced by a subcollection which still forms a base for X.*

2. Relation to ordinary dimension

2.1. THEOREM. *If $\operatorname{Dim} X = n$, then $\dim X \leqslant n$.*

Proof. (By induction.) The result is trivial if $n = -1$; suppose it true if n be replaced by any lesser integer. Choose a base $\{G\}$ for X as in § 1.1, i.e. such that, for each G, $\operatorname{Dim} \mathfrak{B}G/\{G\} \leqslant n-1$. This means that $\operatorname{Dim} \mathfrak{B}G \leqslant n-1$; hence, by the induction hypothesis, $\dim \mathfrak{B}G \leqslant n-1$, and it follows that $\dim X \leqslant n$.

2.2. LEMMA 1. *If $A = \bigcup_{i=1}^{m} A_i$, each A_i is closed, and G is any open set, then*

$$\mathfrak{B}_A(A \cap G) = \bigcup_{i=1}^{m} \mathfrak{B}_{A_i}(A_i \cap G).$$

Proof. Writing $\mathfrak{C}$ for 'complement of', we have

$$\begin{aligned}
\mathfrak{B}_{A_i}(A_i \cap G) &= \overline{A_i \cap G} \cap \mathfrak{C}(A_i \cap G) \\
&= \overline{A_i \cap G} \cap (\mathfrak{C}A_i \cup \mathfrak{C}G) \\
&= (\overline{A_i \cap G} \cap \mathfrak{C}A_i) \cup (\overline{A_i \cap G} \cap \mathfrak{C}G) \\
&= \overline{A_i \cap G} \cap \mathfrak{C}G,
\end{aligned}$$

since $\overline{A_i \cap G} \cap \mathfrak{C}A_i \subset \overline{A_i} \cap \mathfrak{C}A_i = \phi$, A_i being closed. Similarly,

$$\begin{aligned}\mathfrak{B}_A(A \cap G) &= \overline{A \cap G} \cap \mathfrak{C}G \\ &= \overline{\bigcup_{i=1}^{m} A_i \cap G} \cap \mathfrak{C}G \\ &= \Big(\bigcup_{i=1}^{m} \overline{A_i \cap G}\Big) \cap \mathfrak{C}G \\ &= \bigcup_{i=1}^{m} (\overline{A_i \cap G} \cap \mathfrak{C}G) \\ &= \bigcup_{i=1}^{m} \mathfrak{B}_{A_i}(A_i \cap G).\end{aligned}$$

2.3. As in (**2**), let $\mathscr{M}_n^r$ be the set of points in E_n (n-dimensional Euclidean space) with not more than r rational coordinates. Let $\{G\}_n$ denote the base for E_n formed of all parallelotopes defined by relations

$$r_k < x_k < r'_k \quad (k = 1, 2, \ldots, n) \tag{i}$$

with the r_k, r'_k all rational.

2.4. Lemma 2. *Suppose that* $A = \bigcup_{i=1}^{m} A_i$, *where* A_i *consists of all points of* $\mathscr{M}_n^r$ *with coordinates satisfying*

$$x_{l(ij)} = r_{ij} \quad (j = 1, 2, \ldots, s), \tag{ii}$$

where s *is an integer independent of* i ($0 \leqslant s \leqslant r+1$) (*if* $s = 0$, *we define* $A = \mathscr{M}_n^r$), $l(i1)$, $l(i2)$, ..., $l(is)$ *are distinct positive integers between* 1 *and* n, *and the* r_{ij} *are all rational; then*

$$\operatorname{Dim} A/\{G\}_n \leqslant r-s.$$

Proof. (By inverse induction on s.) The result is true if $s = r+1$: for then the equations (ii) state that every point of A has at least $r+1$ rational coordinates; since no point of $\mathscr{M}_n^r$ has more than r rational coordinates, $A = \phi$, and $\operatorname{Dim} A/\{G\}_n = -1 = r-(r+1) = r-s$. Suppose the result true if s be replaced by $s+1$. Let G be the open set defined by equations (i), i.e. a typical set of $\{G\}_n$.

By Lemma 1,

$$\mathfrak{B}_A(A \cap G) = \bigcup_{i=1}^{m} \mathfrak{B}_{A_i}(A_i \cap G).$$

Now $\mathfrak{B}_{A_i}(A_i \cap G)$ either is empty (namely, if $r_{ij} \leqslant r_{l(ij)}$ or $r_{ij} \geqslant r'_{l(ij)}$, for some j) or else is contained in the union of the $2(n-s)$ sets which are defined by (ii) together with one of the $2(n-s)$ equations

$$x_k = r_k, \qquad x_k = r'_k \quad (1 \leqslant k \leqslant n;\ k \neq l(ij) \text{ for any } j).$$

Thus $\mathfrak{B}_A(A \cap G)$ is contained in a set in $\mathscr{M}^r_n$ which is of a similar form to A, but with s replaced by $s+1$; by the induction hypothesis and § 1.2,

$$\operatorname{Dim} \mathfrak{B}_A(A \cap G)/\{G\}_n \leqslant r-s-1,$$

and, since G was a typical set of $\{G\}_n$, it follows that

$$\operatorname{Dim} A/\{G\}_n \leqslant r-s.$$

2.5. Taking $s = 0$ in Lemma 2, we have $\operatorname{Dim} \mathscr{M}^r_n \leqslant r$.

2.6. THEOREM. *If X is separable metric and either of* $\operatorname{Dim} X$, $\dim X$ *exists, then the other exists and has the same value.*

Proof. Looking back at § 2.1, we see that it only remains to show that, if $\dim X = r$, then $\operatorname{Dim} X \leqslant r$. But, if $\dim X = r$, X can be embedded in $\mathscr{M}^r_{2r+1}$ [(2) Theorem V 5], and it follows from § 1.2 and § 2.5 that $\operatorname{Dim} X \leqslant r$.

2.7. THEOREM. *If X is a separable metric space and* $\dim X = r$, *then X has a base of open sets $\{G_i\}$ such that*

$$\dim \bigcap_{j=1}^{k} \mathfrak{B}G_{i_j} \leqslant r-k$$

for any $i_1 < i_2 < \ldots < i_k$, $k \leqslant r+1$. (This is Eilenberg and Otto's Corollary 2.)

Proof. It is first necessary to modify the $\{G\}_n$ of §§ 2.3–2.6 so that no two of the sets of $\{G\}_n$ have the same rational appearing as r_k or r'_k in their definitions by (i). To achieve this, we enumerate the base $\{G\}_n$ as $\{G_i\}$, and then, for $i = 1, 2, \ldots$ in turn, replace $r^{(i)}_k$, $r^{(i)\prime}_k$ $(1 \leqslant k \leqslant n)$ by rational numbers $\rho^{(i)}_k$, $\rho^{(i)\prime}_k$, where

$$r^{(i)}_k \leqslant \rho^{(i)}_k < \tfrac{1}{4}(3r^{(i)}_k + r^{(i)\prime}_k), \qquad \tfrac{1}{4}(r^{(i)}_k + 3r^{(i)\prime}_k) < \rho^{(i)\prime}_k \leqslant r^{(i)}_k$$

and

$$\rho^{(i)}_k \neq \rho^{(j)}_k, \quad \rho^{(i)}_k \neq \rho^{(j)\prime}_k, \quad \rho^{(i)\prime}_k \neq \rho^{(j)}_k, \quad \rho^{(i)\prime}_k \neq \rho^{(j)\prime}_k,$$

for every $j < i$; $k = 1, 2, \ldots, n$.

It is easily seen that the $\{G_i\}$ so modified still form a base for $\mathscr{M}^r_n$. Similar arguments to those in the proof of Lemma 2 now show that for $i_1 < i_2 < \ldots < i_k$,

$$\bigcap_{j=1}^{k} \mathfrak{B}G_{i_j} \subset A,$$

where A is as in Lemma 2 with $s = k$, and consequently $\dim A = n-k$ (using the closed-set sum theorem). This proves the result for $X = \mathscr{M}^r_n$; it follows for any separable metric space by the embedding and monotone theorems.

3. The product theorem

3.1. It is possible to prove the product theorem for uniform dimension (and hence for ordinary dimension in a separable metric space) without using any kind of sum theorem. This is of interest in connexion with the theory of transfinite dimension [(**2**) IV 6]. I have shown elsewhere (**3**) that the closed-set sum theorem does not hold for transfinite dimension, and that in consequence an adaptation of the standard proof of the product theorem gives a weaker result than might be expected. If, however, we use the transfinite generalization of *uniform* dimension, the proof which follows does give us the natural generalization of the product theorem.

3.2. THEOREM. *If* $\operatorname{Dim} X$, $\operatorname{Dim} Y$ *exist and* $X, Y \neq \phi$, *then*

$$\operatorname{Dim}(X, Y) \leqslant \operatorname{Dim} X + \operatorname{Dim} Y.$$

Indeed, if $\operatorname{Dim} X/\{G\} = p$, $\operatorname{Dim} Y/\{H\} = q$, *then*

$$\operatorname{Dim}(X, Y)/\{(G, H)\} \leqslant p+q.$$

Here (X, Y) denotes the topological product of X and Y; $\{(G, H)\}$ is, of course, the base by which the topology of (X, Y) is usually defined.

Proof. The pattern of the proof resembles that of §§ 2.4, 2.5. We prove by induction on r that, if $\operatorname{Dim} X/\{G\}$, $\operatorname{Dim} Y/\{H\}$ exist and $A_1, A_2, \ldots, A_m$ are non-empty closed sets in X, $B_1, B_2, \ldots, B_m$ non-empty closed sets in Y, then

$$\operatorname{Dim} \bigcup_{i=1}^{m} (A_i, B_i)/\{(G, H)\} \leqslant \max_{i=1}^{m} (\operatorname{Dim} A_i/\{G\} + \operatorname{Dim} B_i/\{H\}) = r, \text{ say:} \tag{i}$$

the theorem is then the special case of this with $m = 1, A_1 = X, B_1 = Y$. Note that it follows from the hypothesis and 1.2 that r is finite.

Let us write C for $\bigcup_{i=1}^{m} (A_i, B_i)$; then we have to prove that, for each pair G, H selected from $\{G\}$, $\{H\}$ respectively,

$$\operatorname{Dim} \mathfrak{B}_C(C \cap (G, H))/\{(G, H)\} < r. \tag{ii}$$

Lemma 1 (2.2) gives us

$$\mathfrak{B}_C(C \cap (G, H)) = \bigcup_{i=1}^{m} \mathfrak{B}_{(A_i, B_i)}((A_i, B_i) \cap (G, H)) \subset \bigcup_{i=1}^{m} [(A_i, \mathfrak{B}_{B_i}(B_i \cap H)) \cup (\mathfrak{B}_{A_i}(A_i \cap G), B_i)]. \tag{iii}$$

If $r = 0$, we have, for each i, $\operatorname{Dim} A_i/\{G\} = 0$, $\operatorname{Dim} B_i/\{H\} = 0$, i.e. $\mathfrak{B}_{A_i}(A_i \cap G) = \phi$, $\mathfrak{B}_{B_i}(B_i \cap H) = \phi$, for each G, H; thus, by (iii),

$$\mathfrak{B}_C(C \cap (G, H)) = \phi,$$

and so (ii) is satisfied. Now suppose the result to be true if r be replaced by any lesser integer. We have

$$\operatorname{Dim} \mathfrak{B}_{B_i}(B_i \cap H)/\{H\} < \operatorname{Dim} B_i/\{H\}$$

and

$$\operatorname{Dim} \mathfrak{B}_{A_i}(A_i \cap G)/\{G\} < \operatorname{Dim} A_i/\{G\}:$$

so, by (iii),

$$\mathfrak{B}_C(\mathrm{C} \cap (G, H)) \subset \bigcup_{i=1}^{m'} (A'_i, B'_i),$$

where

$$\max_{i=1}^{m'}(\operatorname{Dim} A'_i/\{G\} + \operatorname{Dim} B'_i/\{H\}) < r.$$

By the induction hypothesis and § 1.2, (ii) holds, and the result follows.

4. Notes

4.1. It has been suggested that 'the result [§ 2.6] is a very trivial consequence of the fact that a separable n-dimensional space can be decomposed into $n+1$ 0-dimensional subsets. It suffices to take for the basis $\{G\}$ the collection of all open sets whose boundaries are disjoint with at least one of the $n+1$ 0-dimensional sets.' It is easy to see that this is not so. Let X be the real line: this is a 1-dimensional space which can be decomposed into two 0-dimensional subsets, consisting respectively of the rationals and irrationals. Let H be the complement of the single point 0, and G the complement of the points 1, $\frac{1}{2}$, $\frac{1}{3}$,..., 0. Evidently G and H are open and belong to the base G suggested; but it is easily seen that

$$\mathfrak{B}_{\mathfrak{B}G}(\mathfrak{B}G \cap H)$$

consists of the single point 0: it follows that

$$\operatorname{Dim} X/\{G\} \not\leqslant 1.$$

4.2. My thanks are due to the referee for drawing my attention to reference (**1**).

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THE CRITICAL LATTICES OF THE OFF-CENTRE HYPERCUBE

By H. G. AP SIMON (*Oxford*)

[Received 26 January 1953]

1. Introduction

WITH a suitable choice of rectangular Cartesian coordinates with origin at the lattice origin, an n-dimensional cube which contains the lattice origin as an interior point may be defined by

$$|x_r - k_r| \leqslant 1 \quad (r = 1, 2, \ldots, n),$$

where $0 \leqslant k_r < 1$. In this note I restrict myself to the case in which all the k_r are non-zero, and call the hypercube in this case *off-centre*; I find its critical determinant and show that it possesses a unique critical lattice.

Since all the k_r are non-zero, we may make the change of variables

$$\frac{x_r}{k_r} = y_r, \qquad \frac{1}{k_r} = c_r \quad (r = 1, 2, \ldots, n);$$

and investigate the hyperbody defined by the inequalities

$$|y_r - 1| \leqslant c_r \quad (r = 1, 2, \ldots, n).$$

Algebraically, the result may be stated as

THEOREM. *If* $\quad y_r = \sum_{s=1}^{n} a_{r,s}\xi_s \quad (r = 1, 2, \ldots, n),$

and c_r $(r = 1, 2, \ldots, n)$ *are given finite constants greater than unity, then there exist integer values of the* ξ, *not all zero, such that*

$$|y_r - 1| \leqslant c_r \quad (r = 1, 2, \ldots, n)$$

if $$\Delta = \|a_{r,s}\| \leqslant \left\{\prod_{r=1}^{n}(c_r - 1)\right\}\left\{1 + 2\sum_{r=1}^{n}\frac{1}{(c_r - 1)}\right\};$$

and this is a best-possible result, with critical forms equivalent to

$$\left.\begin{aligned} y_1 &= (c_1+1)\xi_1 + 2\xi_2 + 2\xi_3 + \ldots + 2\xi_n \\ y_2 &= 2\xi_1 + (c_2+1)\xi_2 + 2\xi_3 + \ldots + 2\xi_n \\ y_3 &= 2\xi_1 + 2\xi_2 + (c_3+1)\xi_3 + \ldots + 2\xi_n \\ &\cdots\cdots\cdots\cdots\cdots \\ y_n &= 2\xi_1 + 2\xi_2 + 2\xi_3 + \ldots + (1+c_n)\xi_n \end{aligned}\right\}. \tag{1}$$

Quart. J. Math. Oxford (2), 4 (1953), 204–9.

2. Definitions

Geometrically the problem is to find the critical lattices of the hyperbody A defined by

$$|y_r-1| \leqslant c_r \quad (r = 1, 2,..., n),$$

and, since, if any point P is a lattice point, then so is P', the reflection of P in the origin, the problem is in all respects the same if we consider instead of A the symmetrical non-convex hyperbody B formed by the union of A with A', where A' is the reflection of A in the origin; i.e. A, A' are defined by

$$\left.\begin{aligned} |y_r-1| &\leqslant c_r \\ |y_r+1| &\leqslant c_r \end{aligned}\right\} \quad (r = 1, 2,..., n) \tag{2}$$

and B as the set of points whose coordinates satisfy one or other of these sets of inequalities.

Let the (convex) hyperbody D be defined by the inequalities

$$\left.\begin{aligned} |y_r| &\leqslant 1+c_r \\ |y_r-y_s| &\leqslant c_r+c_s-2 \end{aligned}\right\} \quad (r, s = 1, 2,..., n; r \neq s). \tag{3}$$

3. It is necessary to show that none of the inequalities defining D is redundant; this is most easily done by giving examples of points not satisfying one of them but satisfying all the rest. Let ϵ be a small positive number satisfying $1+\epsilon < c_r$ $(r = 1, 2,..., n)$. Then it is easily seen that the point whose pth and qth coordinates are $3-c_p-\epsilon$, $1+c_q$, and whose other coordinates are each 2, satisfies all the inequalities defining D with the exception of

$$|y_p-y_q| \leqslant c_p+c_q-2,$$

which it does not satisfy; and that the point whose pth coordinate is $1+c_p+\epsilon$ and whose other coordinates are $1+c_r$ $(r = 1, 2,..., n; r \neq p)$ satisfies all the inequalities defining D with the exception of

$$|y_p| \leqslant 1+c_p,$$

which it does not satisfy. It follows that none of the defining inequalities of D is redundant.

4. D is contained in B

Let the point Y having coordinates $(y_1, y_2,..., y_n)$ be a point of D, so that its coordinates satisfy (3). Suppose that Y is not in A', so that for some integer r $(1 \leqslant r \leqslant n)$ we have

$$|y_r+1| > c_r.$$

Now, by (3), $$-c_r-1 \leqslant y_r \leqslant c_r+1,$$

so that $$c_r-1 \leqslant y_r \leqslant c_r+1. \tag{4}$$

Also, for $s \neq r$, it follows from (3) and (4) that

$$\begin{aligned} 1-c_s &= c_r-1-(c_r+c_s-2) \\ &\leqslant y_r-|y_r-y_s| \leqslant y_s; \end{aligned}$$

and so $$1-c_s \leqslant y_s \leqslant 1+c_s. \tag{5}$$

Hence, from (4) and (5), Y belongs to A if it does not belong to A', and so D is contained in B.

5. Admissibility of the lattice defined by (1)

Let Λ be the lattice defined by (1), which may be written

$$y_r = 2\sum_{s=1}^{n} \xi_s + (c_r-1)\xi_r \quad (r = 1, 2, \ldots, n).$$

Suppose that a lattice point of Λ is an inner point of A, so that there exist integer ξ such that

$$1-c_r < y_r < 1+c_r \quad (r = 1, 2, \ldots, n).$$

If $\sum_{s=1}^{n} \xi_s \leqslant 0$, then

$$1-c_r < 2\sum_{s=1}^{n} \xi_s + (c_r-1)\xi_r,$$

and so $$\xi_r > -1-\frac{2}{c_r-1}\sum_{s=1}^{n} \xi_s > -1.$$

Hence $\xi_r \geqslant 0$, and in this case the only possibility is

$$\xi_1 = \xi_2 = \ldots = \xi_n = 0.$$

If $\sum_{s=1}^{n} \xi_s > 0$, then at least one $\xi \geqslant 1$, say $\xi_t \geqslant 1$; and so

$$2\sum_{s=1}^{n} \xi_s + (c_t-1)\xi_t < c_t+1,$$

i.e. $$\sum_{s=1}^{n} \xi_s < \tfrac{1}{2}\{c_t+1-(c_t-1)\xi_t\} \leqslant 1,$$

which contradicts the hypothesis. Hence no point of Λ other than the origin lies within A, and similarly no point of Λ other than the origin lies within A'. Hence Λ is B-admissible, and also, since D lies within B, Λ is D-admissible.

6. The determinant of Λ

The determinant $d(\Lambda)$ of the lattice Λ is (with a slight temporary change in notation)

$$d(\Lambda_n) = \begin{vmatrix} c_1+1 & 2 & 2 & \dots & 2 \\ 2 & c_2+1 & 2 & \dots & 2 \\ 2 & 2 & c_3+1 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & c_n+1 \end{vmatrix}$$

$$= (c_n-1)\,d(\Lambda_{n-1})+2\prod_{s=1}^{n-1}(c_s-1)$$

$$= \left\{\prod_{r=1}^{n}(c_r-1)\right\}\left\{1+2\sum_{r=1}^{n}\frac{1}{c_r-1}\right\}$$

(as may be verified by an elementary induction).

7. The volume of D

Consider the $n+1$ parallelepipeds P_0 defined by

$$3-c_r \leqslant y_r \leqslant 1+c_r \quad (r = 1, 2,\dots, n);$$

and P_r $(r = 1, 2,\dots, n)$ defined by

$$-c_r-1 \leqslant y_r \leqslant 3-c_r,$$

$$-c_r-c_s+2+2(c_r-1) \leqslant y_s-y_r \leqslant c_r+c_s-2 \qquad (s \neq r;\ s = 1, 2,\dots, n).$$

It is clear that $P_0, P_1,\dots, P_n$ are all contained in D, and that no inner point of P_0 belongs to any of $P_1, P_2,\dots, P_n$.

Further, if Y is an inner point of P_r and $r \neq s$, then

$$c_r-c_s < y_s-y_r,$$

and so Y is not an inner point of P_s. Thus no two of the bodies $P_0, P_1,\dots, P_n$ have an inner point in common. Hence

$$V(D) \geqslant \sum_{r=0}^{n} V(P_r)$$

$$= 2^n\prod_{r=1}^{n}(c_r-1)+\sum_{r=1}^{n}\left\{2^{n+1}\prod_{\substack{s=1\\ s\neq r}}^{n}(c_s-1)\right\}$$

$$= 2^n d(\Lambda).$$

But it has been shown that Λ is an admissible lattice of D, and so it follows, by Minkowski's fundamental theorem, that $V(D) = 2^n d(\Lambda)$ and that Λ is a critical lattice of D (and incidentally that D is a space-filler).

Since D is contained in B and Λ is an admissible lattice of B, it follows

that Λ is a critical lattice of B, and so of A, which proves the first part of the theorem.

8. Uniqueness of Λ

We now have to show that Λ is the only critical lattice of D, from which it will follow immediately that it is the only critical lattice of B and A.

Call any two bounding flats of D *adjacent* if they have in common an $(n-2)$-dimensional boundary of D. From (3) we see that the bounding flats of D adjacent to that lying in

$$y_r = 1+c_r$$

are those lying in

$$y_s = 1+c_s \qquad (s \neq r;\ s = 1, 2,..., n),$$

$$y_r-y_s = c_r+c_s-2 \quad (s \neq r;\ s = 1, 2,..., n);$$

and that the bounding flats adjacent to that lying in

$$y_r-y_s = c_r+c_s-2$$

are those lying in

$$y_r = c_r+1,$$

$$-y_s = c_s+1,$$

$$y_r-y_t = c_r+c_t-2 \quad (t \neq r, s;\ t = 1, 2,..., n),$$

$$y_t-y_s = c_t+c_s-2 \quad (t \neq r, s;\ t = 1, 2,..., n).$$

Hence, if we consider D as being bounded by the flats $\alpha, \beta, \gamma,..., \alpha', \beta', \gamma',...$, where dashes denote reflection in the centre of D, we have

(i) no bounding flat other than α' is parallel to α;
(ii) if β is adjacent to α, then β' is not adjacent to α;
(iii) the internal angle between two adjacent bounding flats is either a right angle or obtuse.

Let D_r $(r = 1, 2,...)$ be hyperbodies parallel and congruent to D, bounded by flats $\alpha_r, \beta_r, \gamma_r,..., \alpha'_r, \beta'_r, \gamma'_r,...$ corresponding to $\alpha, \beta, \gamma,..., \alpha', \beta', \gamma',...$ of D. Arrange the set of D_r to fill space, as we know may be done in at least one way. Then common to at least part of α_1 (of D_1) is at least part of a bounding flat of D_2 (say), and by (i) this is α'_2.

Assume that $\alpha'_2 \neq \alpha_1$. Then common to part of α_1 is part of α'_3 (of D_3), and so on until α_1 has been completely covered by (at least two) flats $\alpha'_2, \alpha'_3,...$ of $D_2, D_3,...$. At least two of $\alpha'_2, \alpha'_3,...$ have an $(n-2)$-dimensional boundary P in common; call these two α'_2, α'_3. Let β_2 be the flat of D_2

adjacent to α_2' at P, and γ_3 the flat of D_3 adjacent to α_3' at P. Then, by (iii), D_2, D_3 intersect unless β_2, γ_3 are (both perpendicular to α_1 and) parallel. But the only flats of D_3 parallel to β_2 are β_3 and β_3', and so γ_3 is β_3'. But β_2 is adjacent to α_2', and so, by (ii), β_3' is not adjacent to α_3', contrary to hypothesis; hence the assumption $\alpha_2' \not\equiv \alpha_1$ is false.

It follows that in a space-filling of D_r the position of each element is uniquely determined, and so the space-filling is unique. But, since, if D possessed a critical lattice other than Λ, there would be a space-filling of D_r other than that corresponding to Λ, it follows that Λ is the unique critical lattice of D, which proves the second part of the theorem.

In conclusion I wish to express my thanks to the referee for several valuable suggestions.

A SCALE OF INTEGRALS FROM LEBESGUE'S TO DENJOY'S

By J. C. BURKILL (*Cambridge*) *and* F. W. GEHRING (*Harvard*)

[Received 2 February 1953]

A SCALE of integrability is known for integrals more general than that of Denjoy (the Cesàro–Perron scale (**3**)). The object of this paper is to set up a scale which spans the gap between the Lebesgue and the (special) Denjoy integrals. We shall define a D_α integral for $0 \leqslant \alpha \leqslant 1$. By definition the D_0 integral is the special Denjoy integral and we shall assume all relevant theorems about this integral and its approach by Perron major and minor functions. The D_1 integral will be shown to be equivalent to the Lebesgue integral. After proving some properties of the D_α integral we illustrate its use by making applications to D_α Fourier series.

Following N. Wiener and L. C. Young (with $1/\alpha$ written for their p for convenience) we shall say for $0 < \alpha \leqslant 1$ that '$F(x)$ is in W_α over the interval $I \equiv (a \leqslant x \leqslant b)$' if

$$V_\alpha\{F(x); x \in I\} = V_\alpha\{F(x); a \leqslant x \leqslant b\}$$
$$= \max\{\sum |\Delta F|^{1/\alpha}\}^\alpha$$

is finite, where $\Delta F = F(x_n) - F(x_{n-1})$ and the sum is taken over any finite dissection of $a \leqslant x \leqslant b$. We define

$$V_0\{F(x); x \in I\} = V_0\{F(x); a \leqslant x \leqslant b\}$$

as the *oscillation* of $F(x)$ over $a \leqslant x \leqslant b$ and W_0 as the *class* of bounded measurable functions defined over this interval.

There is an important subclass of W_α. We say for $0 < \alpha \leqslant 1$ that '$F(x)$ is in V_α over $a \leqslant x \leqslant b$' if, given ϵ, there exists δ such that

$$\{\sum |\Delta F|^{1/\alpha}\}^\alpha < \epsilon$$

for sums taken over any finite set of disjoint intervals for which

$$\{\sum |\Delta x|^{1/\alpha}\}^\alpha < \delta.$$

(We call two closed intervals *disjoint* if they have no common interior points.)

V_0 is defined as the class of functions continuous over $a \leqslant x \leqslant b$ and we see that V_1 is the class of functions absolutely continuous over this interval.

Quart. J. Math. Oxford (2) 4 (1953), 210–20.

DEFINITION 1.1. Suppose that $0 \leqslant \alpha \leqslant 1$. We say that $f(x)$ is 'D_α-integrable over $0 \leqslant x \leqslant 1$' (or, simply, is 'in D_α') if it is Perron-integrable over this interval and has a major function in W_α. We define

$$D_\alpha \int_0^1 f(x)\,dx = P \int_0^1 f(x)\,dx.$$

THEOREM 1.2. *Suppose that* $0 \leqslant \alpha \leqslant 1$, *that* $f(x)$ *is in* D_0, *and that*

$$F(x) = P \int_0^x f(t)\,dt.$$

Then $f(x)$ *is in* D_α *if and only if* $F(x)$ *is in* W_α.

Let $O(x)$ be any major function. There exists a sequence $U_n(x)$ of minor functions which converges uniformly to $F(x)$. We see that

$$O(x) - F(x) = \lim_{n\to\infty}\{O(x) - U_n(x)\}$$

and is hence a non-decreasing function and in W_1. $O(x)$ is in W_α if and only if $F(x)$ is, and the theorem is proved.

As a particular case of this, the major and minor functions of a D_1-integrable function are in W_1 and so the class D_1 is simply that of Lebesgue-integrable functions. There is, however, no connexion between the class D_α for $0 < \alpha < 1$ and the class L_β for $0 < \beta < 1$, i.e. the class of measurable functions for which

$$\int_0^1 |f(x)|^\beta\,dx < \infty,$$

where this integral is taken in the Lebesgue sense. We can prove easily

THEOREM 1.3. (1) *There exists a function* $f(x)$ *in* D_0 *which is in* L_β *for each* $0 < \beta < 1$ *and which is not in* D_α *for any* $0 < \alpha < 1$.

(2) *There exists a function* $f(x)$ *which is in* D_α *for each* $0 < \alpha < 1$ *and which is not in* L_β *for any* $0 < \beta < 1$.

We can obtain a result slightly more precise than Theorem 1.2 if we recall the notions of generalized bounded variation in the restricted sense, and generalized absolute continuity in the restricted sense [(**7**) 228 and 231].

For any finite function $F(x)$ defined over some interval containing the set E we denote by

$$V^*\{F(x);\, x \in E\} \tag{2.1}$$

the upper bound of the sums

$$\sum V_0\{F(x);\, x \in I_n\}, \tag{2.2}$$

where $\sum I_n$ is any finite set of closed disjoint intervals with end-points in E. We say that $F(x)$ is 'VB^* on E' if (2.1) is finite and that $F(x)$ is 'VBG^* on E' if E is the sum of a finite or enumerable sequence of sets on each of which $F(x)$ is VB^*.

Similarly, if $F(x)$ is bounded on an interval containing E, we say that $F(x)$ is 'AC^* on E' if, given ϵ, there exists a δ such that for any finite sequence $\sum I_n$ of closed disjoint intervals whose end-points belong to E, the inequality

$$\sum |I_n| < \delta \quad \text{implies that} \quad \sum V_0\{F(x); x \in I_n\} < \epsilon$$

where $|I_n|$ denotes the length of the interval I_n. $F(x)$ is ACG^* on E if it is continuous on E and if E is expressible as the sum of a finite or enumerable sequence of sets on each of which $F(x)$ is AC^*. We see immediately that a function which is ACG^* on a set is VBG^* on the same set.

Suppose that I is any closed interval which contains points of a set E in its interior. The set of points of E which lie in the interior of I is called a *portion* of the set E. Theorems 2.3 and 2.4 are given by Saks (**8**). Theorem 2.5 follows from Theorems 1.2 and 2.4.

THEOREM 2.3. *In order that a function which is continuous on a closed set E be VBG^* on E, it is necessary and sufficient that every closed subset of E contain a portion on which the function is VB^*.*

THEOREM 2.4. *A necessary and sufficient condition that $F(x)$ be an indefinite integral of a function in D_0 is that $F(x)$ be ACG^* over $0 \leqslant x \leqslant 1$.*

THEOREM 2.5. *Suppose that $0 \leqslant \alpha \leqslant 1$. A necessary and sufficient condition that $F(x)$ be an indefinite integral of a function in D_α is that $F(x)$ be in W_α and ACG^* over $0 \leqslant x \leqslant 1$.*

We have the following result.

THEOREM 2.6. *Suppose that $0 < \alpha < 1$ and that $F(x)$ is continuous, VBG^*, and in W_α over $0 \leqslant x \leqslant 1$. Then $F(x)$ is in V_α over this interval.*

We prove this theorem for two special cases and then use Theorem 2.3 to combine these results and treat the general case.

[2.6.1] *Suppose that $0 \leqslant a < b \leqslant 1$ and that $F(x)$ is in V_α over each interval $c \leqslant x \leqslant d$, where $a < c < d < b$. Then $F(x)$ is in V_α over $a \leqslant x \leqslant b$.*

From [(**12**) 454] we can first choose $a < c < d < b$ so that

$$V_\alpha\{F(x); a \leqslant x \leqslant c\} < \tfrac{1}{6}\epsilon, \qquad V_\alpha\{F(x); d \leqslant x \leqslant b\} < \tfrac{1}{6}\epsilon$$

and then choose δ so that

$$\{\sum |\Delta x|^{1/\alpha}\}^\alpha < \delta \quad \text{implies that } \{\sum |\Delta F|^{1/\alpha}\}^\alpha < \tfrac{1}{3}\epsilon$$

for each set of disjoint intervals lying in the interval $c \leqslant x \leqslant d$. Since $F(x)$ is continuous, we can assume that

$$|\Delta x| < \delta \quad \text{implies that } |\Delta F| < \tfrac{1}{3}\epsilon$$

for each Δx lying in $a \leqslant x \leqslant b$. Minkowski's inequality allows us to conclude that

$$\{\sum |\Delta x|^{1/\alpha}\}^\alpha < \delta \quad \text{implies that } \{\sum |\Delta F|^{1/\alpha}\}^\alpha < \epsilon$$

for each set of disjoint intervals lying in $a \leqslant x \leqslant b$, and hence $F(x)$ is in V_α over this interval.

[2.6.2] *Suppose that* $0 \leqslant a < b \leqslant 1$ *and that* $F(x)$ *is* VB^* *on a closed set* Q *which is contained in the interval* $a \leqslant x \leqslant b$ *and which contains the end-points* $x = a$ *and* $x = b$. *Suppose also that* $F(x)$ *is in* V_α *over each interval contiguous to* Q *relative to* $a \leqslant x \leqslant b$. *Then* $F(x)$ *is in* V_α *over* $a \leqslant x \leqslant b$.

Let $\{I_n\}$ be the sequence of intervals contiguous to Q in $a \leqslant x \leqslant b$. Since $F(x)$ is in W_α over $a \leqslant x \leqslant b$, we can pick N so that

$$\Big\{\sum_{n=N+1}^{\infty} V_\alpha\{F(x);\, x \in I_n\}^{1/\alpha}\Big\}^\alpha < \tfrac{1}{4}\epsilon. \tag{2.6.3}$$

We can pick δ such that

$$\{\sum |\Delta x|^{1/\alpha}\}^\alpha < \delta \text{ implies that } \{\sum |\Delta F|^{1/\alpha}\}^\alpha < \tfrac{1}{4}\epsilon \tag{2.6.4}$$

for each set of disjoint intervals lying in the intervals $I_1, I_2, \ldots, I_N$, and we can also assume that

$$|\Delta F|^{1-\alpha} V^*\{F(x);\, x \in Q\}^\alpha < \epsilon/2^{1+\alpha} \tag{2.6.5}$$

for each $|\Delta x| < \delta$ lying in $a \leqslant x \leqslant b$.

Now take any set of disjoint intervals Δx in $a \leqslant x \leqslant b$ such that

$$\{\sum |\Delta x|^{1/\alpha}\}^\alpha < \delta.$$

We can write

$$\begin{aligned}\{\sum |\Delta F|^{1/\alpha}\}^\alpha &\leqslant \Big\{\sum_1 |\Delta F|^{1/\alpha}\Big\}^\alpha + \Big\{\sum_2 |\Delta F|^{1/\alpha}\Big\}^\alpha \\ &= S_1 + S_2,\end{aligned}$$

where the first sum is taken over all the intervals contained in some contiguous interval. We see from (2.6.3) and (2.6.4) that S_1 is majorized by $\frac{1}{2}\epsilon$ and from (2.6.5) we have

$$S_2 \leqslant \max |\Delta F|^{1-\alpha} \Big\{\sum_2 |\Delta F|\Big\}^\alpha < \tfrac{1}{2}\epsilon.$$

This completes the proof of [2.6.2].

We can now prove Theorem 2.6. Let G be the union of the interiors of all closed intervals lying in $0 \leqslant x \leqslant 1$ over which $F(x)$ is in V_α. $F(x)$ is in V_α over any closed interval which lies in G and by [2.6.1] we need only show that the open interval $0 < x < 1$ lies in G.

Take any closed interval I interior to $0 \leqslant x \leqslant 1$ and suppose that this interval is not contained in G. The set $E = I - GI$ is closed in $0 \leqslant x \leqslant 1$ and by Theorem 2.3 there exists a portion of E over which $F(x)$ is VB^*. Call this portion P, let Q be the closure of P, and let J be the smallest closed interval which contains Q. By [2.6.1] we can assume that P has no isolated points. $F(x)$ is VB^* over Q and in V_α over each interval contiguous to Q. By [2.6.2] this means that $F(x)$ is in V_α over the closed interval J and this gives us a contradiction.

THEOREM 2.7. *Suppose that* $0 \leqslant \alpha \leqslant 1$. *If* $F(x)$ *is an indefinite integral of a* D_α*-integrable function, then* $F(x)$ *is in* V_α.

This is known when $\alpha = 1$ and $\alpha = 0$. When $0 < \alpha < 1$, $F(x)$ is ACG^* and in W_α and the conclusion follows from Theorem 2.6.

We note in passing that Theorem 2.6 still holds if we replace 'VBG^*' by 'VBG' in the hypotheses.

A. J. Ward has extended the Perron definition and defined a 'Perron–Stieltjes' integral which includes the ordinary Perron integral as a special case. We are not concerned with this extension here but we need one theorem (3.1) which is a simple consequence of two of his theorems.

THEOREM 3.1. *Suppose that* $f(x)$ *is in* D_0 *and that* $F(x)$ *is an indefinite integral of* $f(x)$. *If the Stieltjes integral*

$$\int_0^1 G(x)\,dF(x) \tag{3.2}$$

exists in the Riemann–Stieltjes sense, the function $G(x)f(x)$ *is in* D_0 *and*

$$P\int_0^1 G(x)f(x)\,dx = RS\int_0^1 G(x)\,dF(x).$$

THEOREM 3.3. *If* $f(x)$ *is in* D_α *and* $G(x)$ *is in* W_β, *where* $\alpha+\beta > 1$, *then* $G(x)f(x)$ *is in* D_α.

If $F(x)$ is an indefinite integral of $f(x)$, the Stieltjes integral (3.2) exists in the Riemann–Stieltjes sense since $F(x)$ is continuous and $\alpha+\beta > 1$. Hence we need only show that

$$S(x) = P\int_0^x G(t)f(t)\,dt = RS\int_0^x G(t)\,dF(t)$$

is in W_α.

We can assume without loss of generality that

$$|G(x')|+V_\beta\{G(x);\, x' \leqslant x \leqslant x''\} \leqslant 1$$

for each $0 \leqslant x' < x'' \leqslant 1$. From Young's theorem [(11) 266] we get

$$|S(x'')-S(x')| \leqslant K(\alpha,\beta)V_\alpha\{F(x);\, x' \leqslant x \leqslant x''\},$$

and it follows that

$$V_\alpha\{S(x);\, 0 \leqslant x \leqslant 1\} \leqslant K(\alpha,\beta)V_\alpha\{F(x);\, 0 \leqslant x \leqslant 1\}.$$

Theorem 3.3 is not generally true when $\alpha+\beta = 1$ and, if $0 < \alpha < 1$, there exists a function $f(x)$ in D_α and a continuous function $G(x)$ in $W_{1-\alpha}$ such that the function $G(x)f(x)$ is not Perron-integrable over the interval $0 \leqslant x \leqslant 1$. However, when $\alpha = 1$ and $\beta = 0$, it is well known that $G(x)f(x)$ is Lebesgue-integrable and hence in D_1, and similarly, when $\alpha = 0$ and $\beta = 1$, the function $G(x)f(x)$ is Perron-integrable and in D_0.

Theorem 3.3 has the following converse which we state without proof.

THEOREM 3.4. *Suppose that* $0 \leqslant \alpha \leqslant 1$ *and that the function* $G(x)f(x)$ *is in* D_0 *for each* $f(x)$ *in* D_α. *Then* $G(x)$ *is equivalent to a function in* $W_{1-\alpha}$.

We can introduce a metric in the vector space D_α. For $0 \leqslant \alpha \leqslant 1$ we define the distance between any two functions $f(x)$ and $g(x)$ in D_α as

$$\rho_\alpha\{f-g\} = V_\alpha\{F(x)-G(x);\, 0 \leqslant x \leqslant 1\},$$

where $F(x)$ and $G(x)$ are indefinite integrals of the functions $f(x)$ and $g(x)$ respectively. Then $\rho_\alpha\{f-g\}$ is a metric over the space D_α. For D_1 we see that

$$\rho_1\{f-g\} = L\int_0^1 |f(x)-g(x)|\,dx,$$

and this is the metric usually employed in this space.

It is well known that D_1 is complete with the above metric. For $0 \leqslant \alpha < 1$ this is not true and there exists a sequence of functions in D_1 which converge in the D_α metric and which have no limit function in D_0.

THEOREM 4.1. *Suppose that* $0 \leqslant \alpha \leqslant 1$ *and that* $L(f)$ *is a bounded linear functional defined over the metric space* D_α. *There exists a function* $G(x)$ *in* $W_{1-\alpha}$ *such that for each* $f(x)$ *the function* $G(x)f(x)$ *is in* D_0 (*actually in* D_α) *and*

$$L(f) = P\int_0^1 G(x)f(x)\,dx.$$

When $\alpha = 1$, this result is well known [(1) 65]. When $0 \leqslant \alpha < 1$, the linear functional $L(f)$ induces another linear functional $T(F)$ defined over the space of continuous functions in W_α with the norm

$$|F(0)|+V_\alpha\{F(x);\, 0 \leqslant x \leqslant 1\}.$$

We need only define $T(F) = L(f)$ when $F(x)$ is an indefinite integral of the function $f(x)$ in D_α and then use a well-known extension theorem.

If $F(x)$ is in V_α, we can represent this functional as

$$T(F) = \int_0^1 F(x)\, dH(x),$$

where $H(x)$ is in $W_{1-\alpha}$ and where the integral exists in the Riemann–Stieltjes sense. [See (**1**) 61 for the case $\alpha = 0$ and (**6**) 248 for the case $0 < \alpha < 1$.] Write $G(x) = H(1) - H(x)$. Theorem 4.1 then follows from an integration by parts and Theorem 3.1.

5. We conclude with a short discussion of the trigonometric series associated with functions in D_α. We replace the fundamental interval $0 \leqslant x \leqslant 1$ by the interval $0 \leqslant x \leqslant 2\pi$ and assume that all functions have period 2π.

For $0 \leqslant \alpha \leqslant 1$ we say that the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \tag{5.1}$$

is the Fourier series for $f(x)$ in D_α if

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \quad (n = 0, 1, 2,\ldots),$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx \quad (n = 1, 2,\ldots).$$

All these integrals exist in the Perron sense.

THEOREM 5.2. *Suppose that* $0 \leqslant \alpha \leqslant 1$, *that* $f(x)$ *is in* D_α, *and that* (5.1) *is the Fourier series for* $f(x)$. *Then*

$$a_n = o(n^{1-\alpha}), \qquad b_n = o(n^{1-\alpha}).$$

This result is well known when $\alpha = 0, 1$. For $0 < \alpha < 1$ divide the interval $0 \leqslant x \leqslant 2\pi$ into N equal sub-intervals

$$I_n \equiv \{2(n-1)\pi/N \leqslant x \leqslant 2n\pi/N\}.$$

Since

$$\Big\{\sum_{n=1}^{N} |I_n|^{1/\alpha}\Big\}^\alpha = 2\pi N^{\alpha-1},$$

we see that

$$\Big\{\sum_{n=1}^{N} V_0\{F(x);\, x \in I_n\}^{1/\alpha}\Big\}^\alpha < \tfrac{1}{4}\epsilon$$

for $N > N_0$. Furthermore it is easy to show that

$$\Big|\int_{I_n} f(x)\sin Nx\, dx\Big| \leqslant 4V_0\{F(x);\, x \in I_n\}$$

for $n = 1,..., N$. Summing and using Hölder's inequality we get

$$\pi|b_N| \leqslant \sum_{n=1}^{N} \left| \int_{I_n} f(x) \sin Nx \, dx \right| < \epsilon N^{1-\alpha}$$

for $N > N_0$.

We now prove that this theorem is in a sense best-possible.

THEOREM 5.3. *Suppose that $\lambda(n)$ is a positive, decreasing function which tends to 0 as n tends to infinity and suppose that $0 \leqslant \alpha \leqslant 1$. There exists a function $f(x)$ in D_α with at most one point of non-absolute integrability such that*

$$a_n, b_n \neq o\{n^{1-\alpha}\lambda(n)\}.$$

Following an example due to Titchmarsh (9) we construct an odd function for which

$$b_n \sim n^{1-\alpha}\lambda(n)$$

as n tends to infinity through an appropriate sequence of values.

In the interval $I_k \equiv \{\pi/(k+1) \leqslant x \leqslant \pi/k\}$, where $k = 1, 2,...$, define

$$f(x) = k^2 n_k^{1-\alpha} \lambda(n_k) \sin n_k x,$$

where n_k is a sequence of integers such that $n_k > 2n_{k-1}$, n_k is a multiple of $k(k+1)$, and the series

$$\sum_{k=1}^{\infty} k^{2(1-\alpha)} \lambda(n_k)$$

is convergent. It is easy to see that $f(x)$ is in D_0 and, if $F(x)$ is an indefinite integral of $f(x)$, we have

$$V_\alpha\left\{F(x); \frac{\pi}{k+1} \leqslant x \leqslant \frac{\pi}{k}\right\} = 2k^2 n_k^{-\alpha} \lambda(n_k) \left\{\frac{n_k}{k(k+1)}\right\}^\alpha$$

for $k = 1, 2,...$. Since $f(x)$ is odd, we have

$$V_\alpha\{F(x); 0 \leqslant x \leqslant 2\pi\} \leqslant 2 \sum_{k=1}^{\infty} V_\alpha\{F(x); \pi/(k+1) \leqslant x \leqslant \pi/k\} < \infty,$$

and so $f(x)$ is in D_α.

Finally we can write

$$\pi b_{n_j} = \sum_{k=1}^{\infty} 2k^2 n_k^{1-\alpha} \lambda(n_k) \int_{\pi/(k+1)}^{\pi/k} \sin n_k x \sin n_j x \, dx.$$

The jth term of this sum is simply

$$\pi \frac{j^2}{j(j+1)} n_j^{1-\alpha} \lambda(n_j),$$

and it is easy to show that the sum of the remaining terms approaches 0 as j tends to infinity.

THEOREM 5.4. (*Riemann Reduction.*) *Suppose that* $0 \leqslant \alpha < 1$ *and that* $\phi(t)$ *has period* 2π *and is in* D_α. *For each* $\eta > 0$ *and* $\lambda \geqslant 1-\alpha$, *we have, as* $\omega \to \infty$,

$$\omega \int_\eta^\infty \phi(t)\gamma_{1+\lambda}(\omega t)\,dt = o(1),$$

where

$$\gamma_{1+\lambda}(t) = \int_0^1 (1-v)^\lambda \cos vt\,dv.$$

Pollard (7) has proved this result for the case $\alpha = 0$ and we can assume that $0 < \lambda < 1$. We first show that for $T \geqslant \eta$ and $\omega \geqslant \omega(\eta, \epsilon)$ we have

$$\left|\omega \int_T^{T+2\pi} \phi(t)\gamma_{1+\lambda}(\omega t)\,dt\right| < \frac{\epsilon}{T^{1+\lambda}}. \tag{5.4.1}$$

We have the following asymptotic formulae:

$$\gamma_{1+\lambda}(t) = \frac{A\sin(t-\frac{1}{2}\pi\lambda)}{t^{1+\lambda}} + O\left(\frac{1}{t^2}\right), \tag{5.4.2}$$

$$\gamma'_{1+\lambda}(t) = \frac{A\cos(t-\frac{1}{2}\pi\lambda)}{t^{1+\lambda}} + O\left(\frac{1}{t^{2+\lambda}}\right). \tag{5.4.3}$$

If $\omega \geqslant \omega_0(\eta, \epsilon)$, we see from (5.4.2) that there exists a set of numbers $T = \tau_0 < \tau_1 < \ldots < \tau_N = T+2\pi$ such that

for $n = 1,\ldots, N-1$, and

$$\gamma_{1+\lambda}(\omega\tau_n) = 0 \tag{5.4.4}$$

$$\tau_n - \tau_{n-1} < 2\pi/\omega \tag{5.4.5}$$

for $n = 1,\ldots, N$, and such that $N < 3\omega$. Furthermore, if $\Phi(t)$ is an indefinite integral of $\phi(t)$, we can assume that

$$\left\{\sum_{n=1}^N V_0\{\Phi(t);\, \tau_{n-1} \leqslant t \leqslant \tau_n\}^{1/\alpha}\right\}^\alpha < \frac{\epsilon}{6\pi B}, \tag{5.4.6}$$

where $B = |A|$ and A is the constant in (5.4.3), since $\Phi(t)$ is in V_α over $T \leqslant t \leqslant T+2\pi$. With (5.4.4) it is not difficult to show that the integral

$$\int_{\tau_{n-1}}^{\tau_n} \phi(t)\gamma_{1+\lambda}(\omega t)\,dt = \int_{\tau_{n-1}}^{\tau_n} \gamma_{1+\lambda}(\omega t)\,d\Phi(t)$$

is majorized by

$$V_0\{\Phi(t);\, \tau_{n-1} \leqslant t \leqslant \tau_n\} V_1\{\gamma_{1+\lambda}(\omega t);\, \tau_{n-1} \leqslant t \leqslant \tau_n\}$$

for $n = 1,\ldots, N$ and from (5.4.3) and (5.4.5) we see for $\omega \geqslant \omega_1(\eta)$ that

$$V_1\{\gamma_{1+\lambda}(\omega t);\, \tau_{n-1} \leqslant t \leqslant \tau_n\} = \int_{\omega\tau_{n-1}}^{\omega\tau_n} |\gamma_{1+\lambda}(t)|\,dt < \frac{2\pi B}{(\omega T)^{1+\lambda}}$$

for $n = 1,\ldots, N$. Summing on n we get, from Hölder's inequality and (5.4.6),

$$\left|\omega \int_T^{T+2\pi} \phi(t)\gamma_{1+\lambda}(\omega t)\,dt\right| \leqslant \omega \sum_{n=1}^{N} \left|\int_{\tau_{n-1}}^{\tau_n} \phi(t)\gamma_{1+\lambda}(\omega t)\,dt\right|$$

$$\leqslant \frac{2\pi B\omega}{(\omega T)^{1+\lambda}} N^{1-\alpha}\left\{\sum_{n=1}^{N} V_0\{\Phi(t);\, \tau_{n-1} \leqslant t \leqslant \tau_n\}^{1/\alpha}\right\}^{\alpha}$$

$$< \frac{\epsilon}{T^{1+\lambda}}\omega^{1-(\alpha+\lambda)}.$$

From this (5.4.1) follows if we pick $\omega(\eta, \epsilon) = \max\{\omega_0(\eta, \epsilon), \omega_1(\eta), 1\}$.

To complete the proof of Theorem 5.4 we see that for $\omega \geqslant \omega(\eta, \epsilon)$ we can write

$$\left|\omega \int_\eta^\infty \phi(t)\gamma_{1+\lambda}(\omega t)\,dt\right| \leqslant \sum_{k=0}^{\infty}\left|\omega \int_{2k\pi+\eta}^{2(k+1)\pi+\eta} \phi(t)\gamma_{1+\lambda}(\omega t)\,dt\right|$$

$$< \epsilon \sum_{k=0}^{\infty} (2k\pi+\eta)^{-1-\lambda}.$$

THEOREM 5.5. [Bosanquet (**2**)] *Suppose that $\phi(t)$ has period 2π and is in D_0 and suppose that $\phi(t) \to 0$ (C, β) as t approaches 0. Then for each $\eta > 0$ we have*

$$\omega \int_0^\eta \phi(t)\gamma_{1+\lambda}(\omega t)\,dt = o(1)$$

for each $\lambda > \beta \geqslant 0$.

Combining these two results we get

THEOREM 5.6. *Suppose that $0 \leqslant \alpha \leqslant 1$, that $f(x)$ has period 2π and is in D_α, and that*

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\} \to 0 \quad (C, \beta)$$

as t approaches 0. The Fourier series for $f(x)$ is then summable (C, λ) to s for $t = x$ where

(1) $\lambda > \beta$ *if* $\beta \geqslant 1-\alpha$,

(2) $\lambda = 1-\alpha$ *if* $0 \leqslant \beta < 1-\alpha$.

A necessary and sufficient condition that the Fourier series for $f(x)$ be summable (C, λ) to s for $t = x$ is that

$$\omega \int_0^\infty \phi(t)\gamma_{1+\lambda}(\omega t)\,dt = o(1).$$

Hence, when $0 \leqslant \alpha < 1$, Theorem 5.6 follows immediately from Theorems 5.4 and 5.5. When $\alpha = 1$, Theorem 5.6 follows from a result due to Hardy and Littlewood.

We can use Theorem 5.3 to show that Theorem 5.4 is in a sense best-possible since the function $f(x)$ defined in Theorem 5.3 has only one point of non-absolute integrability and hence is (C, δ)-continuous p.p. for each $\delta > 0$. If Theorem 5.4 held for some $0 \leqslant \lambda < 1-\alpha$, then the Fourier series for $f(x)$ would be summable (C, μ) for $\lambda < \mu < 1-\alpha$ and by a theorem due to Lebesgue this would mean that

$$a_n = o(n^\mu), \qquad b_n = o(n^\mu),$$

which is not so.

THEOREM 5.7. *If $f(x)$ is in D_α, where $0 < \alpha \leqslant 1$, its Fourier series can be integrated term by term.*

For $F(x) = \int_0^x f(t)\,dt - \frac{1}{2}a_0 x$ is periodic, continuous, and in W_α, and it is the sum of its Fourier series, which converges uniformly [(**11**) 275]. So, if A_n is the coefficient of $\cos nx$ in its Fourier series,

$$A_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos nx\,dx = -\frac{1}{n\pi}\int_0^{2\pi} f(x)\sin nx\,dx = -\frac{b_n}{n}$$

by integration by parts. This gives the result (which fails for $\alpha = 0$).

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ON THE NOTION OF A FORM IDEAL

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[Received 14 March 1953]

1. Introduction

LET V be an algebraic variety which is defined and irreducible over a ground-field k and which lies in m-dimensional affine space; then to V there corresponds a prime ideal $\mathfrak{p}$ in the polynomial ring $k[X_1, X_2, \ldots, X_m]$. Let us suppose that the origin O is a point of V; then every polynomial $f(X) = f(X_1, X_2, \ldots, X_m)$ which vanishes at every point of V will, in particular, vanish at the origin and so will be without a constant term. Denote by $\bar{f}(X)$ the form composed of the terms of lowest degree which occur in $f(X)$ and by $\bar{\mathfrak{p}}$ the homogeneous ideal which is generated by all such $\bar{f}(X)$. I shall call $\bar{\mathfrak{p}}$ the *local form ideal of V at O*. This ideal acquires importance from the fact that it is, in a certain sense, a local birational invariant, and the main part of this paper will be devoted to elucidating and proving this statement.

From the structure of the local form ideal we can obtain information concerning the structure of V in the neighbourhood of O, but this idea has yet to be developed in detail. As an indication of one type of result that can be obtained I have proved the following: *when the local form ideal is prime, the variety is locally analytically irreducible* (see Theorem 6).

The origin has been used in the above description because it makes the definition of the form ideal simpler to state and easier to appreciate. Actually the basic concept will be that of the form ideal of V at an irreducible subvariety W.

2. An equivalence relation between ideals

It was stated in the introduction that the form ideal of a variety at one of its subvarieties enjoys a kind of birational invariance. In order to formulate this idea precisely I shall introduce an equivalence relation between ideals. The ideals need not be in the same ring, but each of them will be a homogeneous ideal in a polynomial ring $K[X_1, X_2, \ldots, X_m]$ where K is a *fixed* ground-field. The number of indeterminates will, however, vary from one case to another.

By an *H-isomorphism over K* of the polynomial ring $K[X_1, X_2, \ldots, X_m]$ on to the polynomial ring $K[X'_1, X'_2, \ldots, X'_m]$ I shall mean an isomorphic mapping of the first ring on to the second which leaves every element

of K fixed and which maps each X_i into a *linear form* in the X'_j. Thus for an H-isomorphism over K

$$X_i \to \sum_{j=1}^{m} k_{ij} X'_j \quad (k_{ij} \in K).$$

The matrix (k_{ij}) will of necessity be non-singular, and from this it follows that the inverse isomorphism will also be an H-isomorphism over K.

We now make the following definition.

DEFINITION. *Let* $\mathfrak{a}$ *and* $\mathfrak{a}'$ *be homogeneous ideals in the polynomial rings*

$$K[X_1, X_2, \ldots, X_r] \quad \textit{and} \quad K[X'_1, X'_2, \ldots, X'_s].$$

Then I shall say that $\mathfrak{a}$ *and* $\mathfrak{a}'$ *are '*H*-equivalent over* K*' if for a suitable integer* p *(which is not less than either* r *or* s*) there is an* H*-isomorphism over* K *of*

$$K[X_1, \ldots, X_r, X_{r+1}, \ldots, X_p] \textit{ on to } K[X'_1, \ldots, X'_s, X'_{s+1}, \ldots, X'_p]$$

which transforms

$$(\mathfrak{a}, X_{r+1}, \ldots, X_p) \textit{ into } (\mathfrak{a}', X'_{s+1}, \ldots, X'_p).$$

Let us suppose that σ is an H-isomorphism over K of

$$K[X_1, X_2, \ldots, X_p] \text{ on to } K[X'_1, X'_2, \ldots, X'_p]$$

which transforms

$$(\mathfrak{a}, X_{r+1}, \ldots, X_p) \text{ into } (\mathfrak{a}', X'_{s+1}, \ldots, X'_p),$$

and let q be an integer greater than p. Then we can extend σ to an H-isomorphism over K of

$$K[X_1, \ldots, X_p, X_{p+1}, \ldots, X_q] \text{ on to } K[X'_1, \ldots, X'_p, X'_{p+1}, \ldots, X'_q]$$

by making X_{p+i} correspond to X'_{p+i}. This new isomorphism will transform

$$(\mathfrak{a}, X_{r+1}, \ldots, X_p, X_{p+1}, \ldots, X_q) \text{ into } (\mathfrak{a}', X'_{s+1}, \ldots, X'_p, X'_{p+1}, \ldots, X'_q).$$

With the aid of this observation it is an easy matter to see that 'H-equivalence over K' is a transitive relation. It is clearly symmetric and reflexive.

THEOREM 1. *If* $\mathfrak{a}$ *and* $\mathfrak{a}'$ *are homogeneous ideals in the polynomial rings* $K[X_1, X_2, \ldots, X_r]$ *and* $K[X'_1, X'_2, \ldots, X'_s]$*, and if they are* H*-equivalent over* K*, then the residue rings*

$$K[X_1, X_2, \ldots, X_r]/\mathfrak{a} \quad \textit{and} \quad K[X'_1, X'_2, \ldots, X'_s]/\mathfrak{a}'$$

are isomorphic over K*.*

Note that for the purposes of the theorem we must regard K as a subfield of the two residue rings.

Proof. For a suitable value of $p \geqslant \max(r, s)$ there is an H-isomorphism σ of

$$K[X_1, X_2, \ldots, X_p] \text{ on to } K[X_1', X_2', \ldots, X_p']$$

which transforms

$$(\mathfrak{a}, X_{r+1}, \ldots, X_p) \text{ into } (\mathfrak{a}', X_{s+1}', \ldots, X_p').$$

This isomorphism σ determines an isomorphism over K of

$$K[X_1, X_2, \ldots, X_p]/(\mathfrak{a}, X_{r+1}, \ldots, X_p)$$

on to

$$K[X_1', X_2', \ldots, X_p']/(\mathfrak{a}', X_{s+1}', \ldots, X_p').$$

Let $R = K[X_1, X_2, \ldots, X_r]$; then

$$(\mathfrak{a}, X_{r+1}, \ldots, X_p) \cap R = \mathfrak{a}. \tag{A}$$

For an element of $R[X_{r+1}, \ldots, X_p]$, regarded as a polynomial in $X_{r+1}, \ldots, X_p$ with coefficients in R, will belong to $(\mathfrak{a}, X_{r+1}, \ldots, X_p)$ if and only if its 'constant term' is in $\mathfrak{a}$. It follows that an element of R, that is to say a 'constant polynomial', will be in $(\mathfrak{a}, X_{r+1}, \ldots, X_p)$ if and only if it is in $\mathfrak{a}$. This proves (A).

Consider the natural homomorphism of

$$R[X_{r+1}, \ldots, X_p] \text{ on to } R[X_{r+1}, \ldots, X_p]/(\mathfrak{a}, X_{r+1}, \ldots, X_p).$$

In this mapping, R is mapped on to the whole residue ring; consequently, by (A),

$$R[X_{r+1}, \ldots, X_p]/(\mathfrak{a}, X_{r+1}, \ldots, X_p) \quad \text{and} \quad R/\mathfrak{a}$$

are isomorphic and moreover they are isomorphic over K. Thus

$$K[X_1, X_2, \ldots, X_p]/(\mathfrak{a}, X_{r+1}, \ldots, X_p) \quad \text{and} \quad K[X_1, X_2, \ldots, X_r]/\mathfrak{a}$$

are isomorphic over K and so, similarly, are

$$K[X_1', X_2', \ldots, X_p']/(\mathfrak{a}', X_{s+1}', \ldots, X_p') \quad \text{and} \quad K[X_1', X_2', \ldots, X_s']/\mathfrak{a}.$$

This completes the proof.

It will now be clear that two ideals which are H-equivalent to one another will have a great many properties in common; indeed a large number of such properties can be inferred directly from Theorem 1. It would, however, be tedious to embark on an enumeration and I shall content myself, for the present, by noting that two H-equivalent ideals have the same dimension and that, if one is prime, so is the other. These facts follow immediately from Theorem 1 and they will be needed later in this paper.

3. Form ideals in a local ring

Throughout § 3, Q will denote a local ring whose maximal ideal is $\mathfrak{m}$, and K will denote the residue field $Q/\mathfrak{m}$. Suppose that $\mathfrak{m} = (u_1, u_2, \ldots, u_m)$. Let $a \in Q$ $(a \neq 0)$; let $\phi(u) = \phi(u_1, u_2, \ldots, u_m)$ be a form of degree $s \geqslant 0$

in the u_i with coefficients in Q and such that not all these coefficients are in $\mathfrak{m}$; let $\bar{\phi}(X_1, X_2, \ldots, X_m)$ be obtained from ϕ by replacing each u_i by an indeterminate X_i and by replacing each coefficient by its residue modulo $\mathfrak{m}$. This makes $\bar{\phi}(X_1, X_2, \ldots, X_m)$ a non-null form of degree s in $K[X_1, X_2, \ldots, X_m]$. If now

$$a \equiv \phi(u) \pmod{\mathfrak{m}^{s+1}},$$

$\bar{\phi}$ is called a *leading form* of a for the base $(u_1, u_2, \ldots, u_m)$. The leading forms of the zero element for the base $(u_1, u_2, \ldots, u_m)$ are defined in exactly the same way except that we include among them the null form.

Next suppose that $\mathfrak{a}$ is a proper ideal† in Q. The leading forms of the elements in $\mathfrak{a}$ will generate a homogeneous ideal in $K[X_1, X_2, \ldots, X_m]$, which I shall refer to as the *form ideal of* $\mathfrak{a}$ *for the base* $(u_1, u_2, \ldots, u_m)$. By varying the base we obtain varying form ideals for $\mathfrak{a}$. When $(u_1, u_2, \ldots, u_m)$ is a *minimal* base of $\mathfrak{m}$, the associated form ideal is just the *Leitideal* of W. Krull [(**1**) 208]. It is, however, essential for our purposes that we allow more general bases.

Assume that $\bar{\phi}(X_1, X_2, \ldots, X_m)$ is a form of degree s in $K[X_1, X_2, \ldots, X_m]$ and that a form ϕ is obtained from $\bar{\phi}$ by replacing each coefficient by a representative in Q. (The replacement can be done in an arbitrary manner.) Then $\bar{\phi}$ is a leading form, for the base $(u_1, u_2, \ldots, u_m)$, of an element of $\mathfrak{a}$ if and only if

$$a \equiv \phi(u) \pmod{\mathfrak{m}^{s+1}}$$

for at least one element a of $\mathfrak{a}$.

THEOREM 2. *Let* $\mathfrak{a}$ *be a proper ideal in the local ring* Q*; then any two form ideals of* $\mathfrak{a}$ *are H-equivalent over the residue field* $K = Q/\mathfrak{m}$.

Proof. If a is an element of Q, I shall use $\bar{a}$ to denote its residue modulo $\mathfrak{m}$. The proof will be accomplished in two steps.

First Step.‡ Let $(u_1, u_2, \ldots, u_r)$ and $(v_1, v_2, \ldots, v_r)$ be two *minimal* bases§ of $\mathfrak{m}$. (Here we have made use of the known fact that two minimal bases necessarily contain the same number of elements.) Then

$$u_i = \sum_{j=1}^{r} c_{ij} v_j,$$

† By a *proper ideal* is meant an ideal which is not the whole ring.

‡ See (**1**) 209.

§ $(u_1, u_2, \ldots, u_r)$ is a minimal base of $\mathfrak{m}$ if the u_i generate $\mathfrak{m}$ but no proper subset of the u_i have this property.

where the determinant $|\bar{c}_{ij}|$ is a non-zero element of K. Denote by σ the H-isomorphism over K of

$$K[X_1, X_2, \ldots, X_r] \text{ on to } K[X_1', X_2', \ldots, X_r']$$

which is defined by $$X_i^\sigma = \sum_{j=1}^{r} \bar{c}_{ij} X_j'.$$

Let $\bar{\phi}(X_1, X_2, \ldots, X_r)$ be a form of degree s in $K[X_1, X_2, \ldots, X_r]$ and let $\bar{\psi}(X_1', X_2', \ldots, X_r')$ be its image under σ so that

$$\bar{\phi}(\sum \bar{c}_{1j} X_j', \sum \bar{c}_{2j} X_j', \ldots, \sum \bar{c}_{rj} X_j') = \bar{\psi}(X_1', X_2', \ldots, X_r').$$

Now derive ϕ from $\bar{\phi}$ by replacing each coefficient by a representative from Q. We then have

$$\phi(u_1, u_2, \ldots, u_r) = \phi(\sum c_{1j} v_j, \sum c_{2j} v_j, \ldots, \sum c_{rj} v_j) = \psi(v_1, v_2, \ldots, v_r),$$

where ψ is the form of degree s which is obtained from ϕ in the obvious manner. Note that, if we replace v_j by X_j' and the coefficients of ψ by their residues modulo $\mathfrak{m}$, we obtain the form $\bar{\psi} = \bar{\phi}^\sigma$. But $\bar{\phi}$ is a leading form, for the base (u), of an element of $\mathfrak{a}$ if and only if $a \equiv \phi(u) \pmod{\mathfrak{m}^{s+1}}$ for at least one element a of $\mathfrak{a}$. Since, however, $\phi(u) = \psi(v)$, this shows that $\bar{\phi}$ is a leading form, for the base (u), of an element of $\mathfrak{a}$ if and only if $\bar{\phi}^\sigma$ is a leading form, for the base (v), of an element of $\mathfrak{a}$. It follows that the two form ideals of $\mathfrak{a}$ which correspond to the bases (u) and (v) are H-equivalent over K.

Second Step. Let $(u_1, u_2, \ldots, u_m)$ be an arbitrary base of $\mathfrak{m}$. Without loss of generality we can suppose that $(u_1, u_2, \ldots, u_r)$, where $r \leqslant m$, is a minimal base of $\mathfrak{m}$. Let $\bar{\mathfrak{a}}$ be the form ideal of $\mathfrak{a}$ for the base $(u_1, u_2, \ldots, u_r)$ and $\bar{\mathfrak{a}}'$ the form ideal for the base $(u_1, u_2, \ldots, u_m)$. After what has just been proved it will be sufficient to show that $\bar{\mathfrak{a}}'$ and $\bar{\mathfrak{a}}$ are H-equivalent over K. The ideal $\bar{\mathfrak{a}}$ is in the ring $K[X_1, X_2, \ldots, X_r]$ while $\bar{\mathfrak{a}}'$ is in

$$K[X_1, \ldots, X_r, X_{r+1}, \ldots, X_m].$$

Clearly $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{a}}'$.

Since $(u_1, u_2, \ldots, u_r) = (u_1, u_2, \ldots, u_m)$, we see that for $1 \leqslant i \leqslant m-r$ we have

$$u_{r+i} = \sum_{j=1}^{r} b_{ij} u_j$$

for suitable elements b_{ij} in Q. Put

$$Y_i = X_{r+i} - \sum_{j=1}^{r} \bar{b}_{ij} X_j.$$

Then Y_i is a leading form of the zero element for the larger base and therefore $Y_i \in \bar{\mathfrak{a}}'$. It follows that

$$(\bar{\mathfrak{a}}, Y_1, Y_2, \ldots, Y_{m-r}) \subseteq \bar{\mathfrak{a}}'.$$

I shall show that in fact

$$(\bar{\mathfrak{a}}, Y_1, Y_2, \ldots, Y_{m-r}) = \bar{\mathfrak{a}}'. \tag{B}$$

Assume that $a \equiv \phi(u_1, u_2, \ldots, u_m) \pmod{\mathfrak{m}^{s+1}}$,

where ϕ is a form of degree s and where $a \in \mathfrak{a}$. Then (B) will follow if we prove that

$$\bar{\phi}(X_1, X_2, \ldots, X_m) \in (\bar{\mathfrak{a}}, Y_1, Y_2, \ldots, Y_{m-r}). \tag{C}$$

Now

$$\begin{aligned}\bar{\phi}(X_1, X_2, \ldots, X_m) &= \bar{\phi}(X_1, \ldots, X_r, \textstyle\sum \bar{b}_{1j} X_j + Y_1, \sum \bar{b}_{2j} X_j + Y_2, \ldots) \\ &\equiv \bar{\phi}(X_1, \ldots, X_r, \textstyle\sum \bar{b}_{1j} X_j, \sum \bar{b}_{2j} X_j, \ldots), \quad \text{modulo}(Y_1, Y_2, \ldots, Y_{m-r}).\end{aligned}$$

Again $a \equiv \phi(u_1, \ldots, u_r, \sum b_{1j} u_j, \sum b_{2j} u_j, \ldots) \pmod{\mathfrak{m}^{s+1}}$,

and this shows that

$$\bar{\phi}(X_1, \ldots, X_r, \textstyle\sum \bar{b}_{1j} X_j, \sum \bar{b}_{2j} X_j, \ldots)$$

belongs to $\bar{\mathfrak{a}}$. (C) is therefore established and (B) follows in the manner explained.

If we regard $\bar{\mathfrak{a}}$ as a homogeneous ideal in $K[X_1, X_2, \ldots, X_r]$ and regard $(\bar{\mathfrak{a}}, Y_1, \ldots, Y_{m-r})$ as a homogeneous ideal in the polynomial ring

$$K[X_1, \ldots, X_r, Y_1, \ldots, Y_{m-r}],$$

then the two are clearly H-equivalent over K. But $(\bar{\mathfrak{a}}, Y_1, \ldots, Y_{m-r})$ as a homogeneous ideal in

$$K[X_1, \ldots, X_r, Y_1, \ldots, Y_{m-r}]$$

is H-equivalent to $\bar{\mathfrak{a}}'$, where $\bar{\mathfrak{a}}'$ is regarded as a homogeneous ideal in $K[X_1, X_2, \ldots, X_m]$.† Combining these observations we see that $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{a}}'$ are H-equivalent over K and this completes the proof.

Now that Theorem 2 has been established it will be reasonable to speak of *the* form ideal of an ideal (in a local ring) whenever we have to deal with a property which is not affected by an H-equivalence. For example, we may say that the form ideal has a definite dimension because, as we have seen, two H-equivalent ideals necessarily have the same dimension. In this way, the following important theorem of Krull [(1) 210, Theorem 8] takes on a wider significance in view of our somewhat more general definitions.

THEOREM (*Krull*). *If $\mathfrak{a}$ is a proper ideal in a local ring Q, then $\mathfrak{a}$ and its form ideal have the same dimension.*

4. Some basic properties of form ideals

In this same section we shall have occasion to consider two or more local rings at the same time, but, whenever this situation arises, the local rings being discussed will all have the same residue field. As before,

† To see this it is sufficient to apply the identity mapping.

I shall use Q to denote a local ring and I shall attach to the symbol Q suitable distinguishing marks when this is necessary. The same distinguishing marks will be attached to the maximal ideal $\mathfrak{m}$. Thus in Theorem 3, where we are concerned with rings Q and Q', we automatically use $\mathfrak{m}$ and $\mathfrak{m}'$ to denote their respective maximal ideals. I continue to denote the common residue field by K.

THEOREM 3. *Let σ be a homomorphism of a local ring Q on to a local ring Q' and let $\mathfrak{a}$ and $\mathfrak{a}'$ be (proper) corresponding ideals of these rings so that $\mathfrak{a}' = \sigma(\mathfrak{a})$ and $\mathfrak{a} = \sigma^{-1}(\mathfrak{a}')$. Then each form ideal of $\mathfrak{a}$ is H-equivalent over $K = Q/\mathfrak{m} = Q'/\mathfrak{m}'$ to each form ideal of $\mathfrak{a}'$.*

It is in the proof of this theorem that the advantage of using non-minimal bases, when defining form ideals, becomes apparent.

Proof. Let $\mathfrak{m} = (u_1, u_2, \ldots, u_r)$; then $\mathfrak{m}' = (u_1', u_2', \ldots, u_r')$, where

$$u_i' = \sigma(u_i).$$

I shall now show that the form ideal of $\mathfrak{a}$ for the base (u) is identical with the form ideal of $\mathfrak{a}'$ for the base (u'). This will establish the theorem. Suppose that $\bar{\phi}(X) = \bar{\phi}(X_1, X_2, \ldots, X_r)$ is a form of degree s in

$$K[X_1, X_2, \ldots, X_r].$$

Let ϕ be obtained from $\bar{\phi}$ by choosing, in Q, representatives for the coefficients of $\bar{\phi}$ and let ϕ' be obtained from ϕ by applying σ to these representatives. When this has been done, $\bar{\phi}$ can be obtained from ϕ' by reading its coefficients modulo $\mathfrak{m}'$.

If now $a \equiv \phi(u) \pmod{\mathfrak{m}^{s+1}}$, where $a \in \mathfrak{a}$, then

$$\sigma(a) \equiv \phi'(u') \pmod{\mathfrak{m}'^{s+1}}.$$

This shows that, if $\bar{\phi}$ is a leading form of an element of $\mathfrak{a}$, then it is also a leading form of an element of $\mathfrak{a}'$.

To establish the converse, we assume that $a' \equiv \phi'(u') \pmod{\mathfrak{m}'^{s+1}}$, where $a' \in \mathfrak{a}'$. Choose $a \in \mathfrak{a}$ and $b \in \mathfrak{m}^{s+1}$ so that

$$\sigma(a) = a' \quad \text{and} \quad \sigma(b) = a' - \phi'(u') = \sigma[a - \phi(u)].$$

Then $b - a + \phi(u) = a_1$ (say) belongs to the kernel of the homomorphism σ, so that *a fortiori* $a_1 \in \mathfrak{a}$. We now have

$$a + a_1 \equiv \phi(u) \pmod{\mathfrak{m}^{s+1}},$$

which shows that, if $\bar{\phi}$ is a leading form of an element of $\mathfrak{a}'$, then it is also a leading form of an element of $\mathfrak{a}$. The proof is now complete.

THEOREM 4. *Let $\mathfrak{a}$ be a proper ideal in Q, let Q^* be the completion of Q, and let $\mathfrak{a}^* = Q^*\mathfrak{a}$. Then each form ideal of $\mathfrak{a}$ is H-equivalent over*

$$K = Q/\mathfrak{m} = Q^*/\mathfrak{m}^*$$

to each form ideal of $\mathfrak{a}^*$.

Let $\mathfrak{m} = Qu_1 + Qu_2 + \ldots + Qu_r$. Then

$$\mathfrak{m}^* = Q^*\mathfrak{m} = Q^*u_1 + Q^*u_2 + \ldots + Q^*u_r.$$

It is a simple matter to show that the form ideal of $\mathfrak{a}$ for the base (u) is the same as the form ideal of $\mathfrak{a}^*$ for the base (u), and from this the theorem follows. The details are left to the reader.

Let us recall that the property of being a homogeneous prime ideal is one which is not destroyed by an H-equivalence. From this it follows that, if $\mathfrak{a}$ is an ideal in a local ring, then the assertion that *the form ideal of $\mathfrak{a}$ is prime* has a precise meaning, which is independent of the base with respect to which the form ideal happens to be constructed. We take advantage of this observation in

THEOREM 5. *Let $\mathfrak{a}$ be a proper ideal in a local ring Q and suppose that the form ideal of $\mathfrak{a}$ is prime. Then $\mathfrak{a}$ is itself a prime ideal. Further, the completion of the local ring $Q/\mathfrak{a}$ is an integral domain.*

Proof. Let $\bar{\mathfrak{a}}$ be the form ideal of $\mathfrak{a}$ for a particular minimal base $(u_1, u_2, \ldots, u_r)$ of $\mathfrak{m}$. If the dimension of $\bar{\mathfrak{a}}$ is d, then, by Krull's theorem, that of $\mathfrak{a}$ is also d. We can therefore find a d-dimensional prime ideal $\mathfrak{p}$ which belongs to $\mathfrak{a}$ and then, if $\bar{\mathfrak{p}}$ is the form ideal of $\mathfrak{p}$ for the base $(u_1, u_2, \ldots, u_r)$, $\bar{\mathfrak{p}}$ will be yet another d-dimensional ideal. Since $\mathfrak{p} \supseteq \mathfrak{a}$, we have $\bar{\mathfrak{p}} \supseteq \bar{\mathfrak{a}}$. But $\bar{\mathfrak{a}}$ is prime (by hypothesis) and $\bar{\mathfrak{p}}$ and $\bar{\mathfrak{a}}$ have the same dimension. Consequently $\bar{\mathfrak{p}} = \bar{\mathfrak{a}}$. Now from $\mathfrak{p} \supseteq \mathfrak{a}$ and $\bar{\mathfrak{p}} = \bar{\mathfrak{a}}$ it follows [(1) 208, Theorem 3] that $\mathfrak{p} = \mathfrak{a}$. This proves that $\mathfrak{a}$ is prime and establishes the first assertion of the theorem.

Let Q^* be the completion of Q and let $\mathfrak{a}^* = Q^*\mathfrak{a}$. By Theorem 4, $\mathfrak{a}^*$ and $\mathfrak{a}$ have the same form ideals. Consequently the form ideal of $\mathfrak{a}^*$ is prime. It follows now that $\mathfrak{a}^*$ is a prime ideal and that $Q^*/\mathfrak{a}^*$ is an integral domain. To complete the proof we have only to note that $Q^*/\mathfrak{a}^*$ is, in fact, the completion of $Q/\mathfrak{a}$.

5. Local form ideals in geometry

I shall now examine, very briefly, how these results apply to the local rings which occur in algebraic geometry.

Let V^r be an r-dimensional variety which is defined and irreducible over a ground-field k and which lies in an m-dimensional affine space S^m.

Then to V there corresponds a certain prime ideal $\mathfrak{p}$ in the polynomial ring $k[X_1, X_2, \ldots, X_m]$. Now suppose that W^s is an irreducible† s-dimensional subvariety of V and denote by $Q(W/S^m)$ the local ring of S^m at W. The residue field of $Q(W/S^m)$ is then the field $R(W)$ of rational functions on W. In $Q(W/S^m)$, the ideal $Q(W/S^m)\mathfrak{p}$ is a prime ideal. I shall denote the form ideal of $Q(W/S^m)\mathfrak{p}$ by $\mathfrak{f}(W/V)$ and I shall call $\mathfrak{f}(W/V)$ *the local form ideal of V at W*. Note that $\mathfrak{f}(W/V)$ is a homogeneous ideal which is defined up to an H-equivalence over the field $R(W)$. Again, if $Q(W/V)$ denotes the local ring of V at W, then

$$Q(W/V) = Q(W/S^m)/Q(W/S^m)\mathfrak{p}.$$

Consequently, by Theorem 3, $\mathfrak{f}(W/V)$ *can also be regarded as the form ideal of the zero ideal of* $Q(W/V)$. It is for this reason that the symbol $\mathfrak{f}(W/V)$ contains no reference to the ambient space S^m, and it is for the same reason that we may describe $\mathfrak{f}(W/V)$ as a *local birational invariant*. By Krull's theorem, *the dimension of* $\mathfrak{f}(W/V)$ *is the difference between the dimensions of V and W*. The structure of $\mathfrak{f}(W/V)$ reveals some of the cruder local properties of V at W. As an example of this let us note that, from Theorem 5, we at once obtain

THEOREM 6. *If the local form ideal of V at W is a prime ideal, then V is analytically irreducible at W.*

Finally, if W is the origin O of the affine space, so that V is a variety passing through the origin, then we have the situation which was described in the introduction. In these circumstances, the procedure for obtaining the form ideal can be considerably simplified and it was the simplified version which I described when introducing our subject. It is, however, an easy matter to see that the ideal so obtained is identical with that which appears as the result of applying the definitions of the general theory. The details of the verification are left to the reader.

† The irreducibility of W refers, of course, to the same ground-field k.

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A SQUARE AS THE SUM OF 7 SQUARES

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[Received 19 March 1953]

1. FORMULAE have been given by Hurwitz (**1**) for the numbers of ways a square can be expressed as the sum of 3 and 5 squares; in this paper is given the corresponding result for 7 squares.

Let us write

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)^a = 1+r_a(1)q+r_a(2)q^2+r_a(3)q^3+\ldots,$$

and thus denote by $r_a(n)$ the number of ways n is the sum of a squares. Also, for convenience, let us write

$$\frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\ldots = \frac{(1|4)}{1^s}+\frac{(2|4)}{2^s}+\frac{(3|4)}{3^s}+\frac{(4|4)}{4^s}+\ldots,$$

and note here that $(mn|4) = (m|4)(n|4)$,

and that therefore

$$\zeta(s) = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots = 1\Big/\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\ldots,$$

$$\eta(s) = \frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\ldots$$

$$= 1\Big/\left(1-\frac{(2|4)}{2^s}\right)\left(1-\frac{(3|4)}{3^s}\right)\left(1-\frac{(5|4)}{5^s}\right)\left(1-\frac{(7|4)}{7^s}\right)\ldots,$$

the products being over all primes.

Particular cases of Hurwitz's and our formulae are

$$r_3(p^2) = 6(p+1-(p|4)),$$
$$r_5(p^2) = 10(p^3+1-p),$$
$$r_7(p^2) = 14(p^5+1-p^2(p|4)),$$

where p denotes an odd prime. In the general case, denoting by $\sigma_a(x)$ the sum of the ath powers of the divisors of x, and by $\mu(x)$ the xth Möbius coefficient, and making the convention that the arithmetical functions of x defined when x is an integer are zero when x is not an integer, we have

$$r_3(m^2) = 6\sum_{r\ \text{odd}}\left\{\sigma_1\left(\frac{m}{r}\right)-2\sigma_1\left(\frac{m}{2r}\right)\right\}(r|4)\mu(r),$$

$$r_5(m^2) = 10\sum_{r\ \text{odd}}\sigma_3\left(\frac{m}{r}\right)r\mu(r),$$

$$r_7(m^2) = 14\sum_{r\ \text{odd}}\left\{\sigma_5\left(\frac{m}{r}\right)+8\sigma_5\left(\frac{m}{2r}\right)\right\}r^2(r|4)\mu(r).$$

From these latter general formulae it follows at once that

$$\frac{1}{6}\sum_1^\infty \frac{r_3(m^2)}{m^s} = \frac{\zeta(s)\zeta(s-1)}{\eta(s)}(1-2/2^s) = \frac{1}{1-1/2^s}\prod\prod \frac{1-(p|4)/p^s}{(1-1/p^s)(1-p/p^s)},$$

$$\frac{1}{10}\sum_1^\infty \frac{r_5(m^2)}{m^s} = \frac{\zeta(s)\zeta(s-3)}{\zeta(s-1)}\frac{1}{(1-2/2^s)}$$

$$= \frac{1}{(1-1/2^s)(1-2^3/2^s)}\prod\prod \frac{1-p/p^s}{(1-1/p^s)(1-p^3/p^s)},$$

$$\frac{1}{14}\sum_1^\infty \frac{r_7(m^2)}{m^s} = \frac{\zeta(s)\zeta(s-5)}{\eta(s-2)}(1+2/2^s)$$

$$= \frac{1}{(1-1/2^s)(1-2^5/2^s)}\prod\prod \frac{1-p^2(p|4)/p^s}{(1-1/p^s)(1-p^5/p^s)},$$

where p denotes an odd prime. Hence, if the expression of m in terms of its prime factors is

$$m = 2^\alpha \prod p^\beta,$$

Hurwitz's and our general formulae can be written

$$r_3(m^2) = 6\prod\prod\left\{\frac{p^{\beta+1}-1-(p|4)(p^\beta-1)}{p-1}\right\},$$

$$r_5(m^2) = 10\frac{2^{3\alpha+3}-1}{2^3-1}\prod\prod\left\{\frac{p^{3\beta+3}-1-p(p^{3\beta}-1)}{p^3-1}\right\},$$

$$r_7(m^2) = 14\frac{2^{5\alpha+5}-1+2^3(2^{5\alpha}-1)}{2^5-1}\prod\prod\left\{\frac{p^{5\beta+5}-1-p^2(p|4)(p^{5\beta}-1)}{p^5-1}\right\}.$$

When there are more than 8 squares the number of representations requires arithmetical functions of the type first studied by Glaisher (**2**), and later by Ramanujan (**3**). In a paper submitted to the London Mathematical Society the writer has given formulae in terms of three of these for the numbers of ways a square can be expressed as the sum of 9, 11, and 13 squares. For example, the number of ways the square of an odd prime p can be expressed as the sum of 13 squares is given by

$$r_{13}(p^2) = \frac{4030}{691}(p^{11}+1)-26p^5+\frac{13936}{691}\tau(p),$$

and the general formula is

$$r_{13}(m^2) = \sum_{r\,\text{odd}}\left\{\frac{4030}{691}\sigma_{11}\left(\frac{m}{r}\right)+\frac{13936}{691}\tau\left(\frac{m}{r}\right)\right\}r^5\mu(r),$$

where $\tau(n)$ is defined [(**3**) 151] by

$$q\prod_1^\infty (1-q^n)^{24} = \sum_1^\infty \tau(n)q^n.$$

2. The proof is an application to identities derived from elliptic functions of an elementary theorem, based on the following remark.

If numbers $N(1)$, $N(2)$, $N(3)$,... possess the multiplicative property that

$$N(mn) = N(m)N(n),$$

then

$$N(m^2)+2N(m^2-1^2)+2N(m^2-2^2)+2N(m^2-3^2)+\ldots$$
$$= N^2(m)+2N(m-1)N(m+1)+2N(m-2)N(m+2)+$$
$$+2N(m-3)N(m+3)+\ldots,$$

and therefore the coefficient of q^{m^2} in

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)\{N(1)q+N(2)q^2+N(3)q^3+\ldots\}$$

is equal to the coefficient of q^{2m} in

$$\{N(1)q+N(2)q^2+N(3)q^3+\ldots\}^2.$$

More generally our elementary theorem is that, if numbers $N(1)$, $N(2)$, $N(3)$,... possess the property that for p prime

$$N(pn) = N(p)N(n)-f(p)N(n/p),$$

where

$$f(rs) = f(r)f(s),$$

so that the simple multiplicative property above is replaced by

$$N(mn) = \sum_r N\left(\frac{m}{r}\right)N\left(\frac{n}{r}\right)f(r)\mu(r),$$

and if

either $N(0) = 0$, or $N(0)$ is arbitrary and $N(p) = 1+f(p)$,

then the coefficient of q^{m^2} in

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)\{N(0)+N(1)q+N(2)q^2+\ldots\}$$

is equal to

$$\sum M\left(\frac{2m}{r}\right)f(r)\mu(r),$$

where

$$\{N(0)+N(1)q+N(2)q^2+\ldots\}^2 = M(0)+M(1)q+M(2)q^2+\ldots.$$

For the coefficient is equal to

$$\sum_r \left\{N^2\left(\frac{m}{r}\right)+2N\left(\frac{m-1}{r}\right)N\left(\frac{m+1}{r}\right)+2N\left(\frac{m-2}{r}\right)N\left(\frac{m+2}{r}\right)+\ldots+\right.$$
$$\left.+2N\left(\frac{m-m}{r}\right)N\left(\frac{m+m}{r}\right)\right\}f(r)\mu(r)+2N(0)\left\{1-\sum_r N\left(\frac{2m}{r}\right)f(r)\mu(r)\right\}.$$

And from the multiplicative formula it follows that

$$\frac{N(p)}{1^s}+\frac{N(2p)}{2^s}+\frac{N(3p)}{3^s}+\ldots = \left\{N(p)-\frac{f(p)}{p^s}\right\}\left\{\frac{N(1)}{1^s}+\frac{N(2)}{2^s}+\frac{N(3)}{3^s}+\ldots\right\},$$

where p is prime. Repeated application of this yields

$$\sum_1^\infty \frac{N(n)}{n^s} = \prod \mathrm{1}\Big/\left(1-\frac{N(p)}{p^s}+\frac{f(p)}{p^{2s}}\right),$$

where the product is over all primes. When $N(p) = 1+f(p)$, the above becomes

$$\sum_1^\infty \frac{N(n)}{n^s} = \prod \mathrm{1}\Big/\left(1-\frac{1}{p^s}\right)\left(1-\frac{f(p)}{p^s}\right),$$

whence

$$\sum N\left(\frac{n}{r}\right)f(r)\mu(r) = 0$$

for any n. Hence it follows that the second term of the coefficient of q^{m^2} vanishes, either when $N(0) = 0$ or when $N(p) = 1+f(p)$ with $N(0)$ arbitrary.

3. In deducing his formulae for 3 and 5 squares Hurwitz makes implicit use of this theorem in the cases where $N(0)$, $N(1)$, $N(2)$,... are the coefficients in Jacobi's classical expansions of

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)^2, \quad (1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)^4,$$

by expressing the coefficients of q^{m^2} when these expressions are multiplied by

$$1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots$$

in terms of the coefficients $M(2m)$ of q^{2m} in the expansions of

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)^4, \quad (1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)^8,$$

respectively, and thus deduces $r_3(m^2)$ from the classical formulae for $r_2(m)$ and $r_4(2m)$, and $r_5(m^2)$ from those for $r_4(m)$ and $r_8(2m)$.

4. To establish the main result let us begin by noting the classical identity (4)

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)^6 = 16\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2+1-4\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(-1)^n(2n+1)^2.$$

Then we shall examine the coefficients in the expansions of each of the summations on the right-hand side, and show that we can apply our theorem to connect the coefficients of q^{m^2} in

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2,$$

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)\left(-\frac{1}{4}+\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(-1)^n(2n+1)^2\right),$$

respectively with the coefficients of q^{2m} in

$$\left\{\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2\right\}^2, \qquad \left\{-\frac{1}{4}+\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(-1)^n(2n+1)^2\right\}^2.$$

Finally we prove the identity

$$16\left\{\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2\right\}^2+16\left\{\sum_1^\infty \frac{(-q)^n}{1+q^{2n}}n^2\right\}^2-$$
$$-4\left\{-\frac{1}{4}+\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(-1)^n(2n+1)^2\right\}^2-$$
$$-4\left\{-\frac{1}{4}-\sum_0^\infty \frac{q^{2n+1}}{1+q^{2n+1}}(-1)^n(2n+1)^2\right\}^2$$
$$=-\frac{1}{18}\left\{1-504\sum_1^\infty \frac{q^{2n}}{1-q^{2n}}n^5\right\}-\frac{4}{9}\left\{1-504\sum_1^\infty \frac{q^{4n}}{1-q^{4n}}\right\},$$

and thus connect $r_7(m^2)$ with $\sigma_5(m/r)$.

5. Equating coefficients of x^2 in the power-series and partial-fraction expansions of $\mathrm{dn}\,x/\mathrm{cn}\,x$ and $\mathrm{dn}\,x$, we have

$$\left(\frac{2K}{\pi}\right)^3 k^2 = 16\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2,$$

$$\left(\frac{2K}{\pi}\right)^3 (1-k^2) = 1-4\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(-1)^n(2n+1)^2,$$

whence, of course,

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)^6$$
$$=16\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2+1-4\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(-1)^n(2n+1)^2.$$

We now examine the coefficients in the expansions of the summations of the right-hand side. Writing

$$\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2=\sum_1^\infty A(n)q^n, \qquad \sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(-1)^n(2n+1)^2=\sum_1^\infty B(n)q^n,$$

we have $\quad A(n)=\sum_r r^2\left(\frac{n}{r}\middle|4\right), \qquad B(n)=\sum_r \frac{n^2}{r^2}\left(\frac{n}{r}\middle|4\right).$

Hence, if p is prime,

$$A(pn)=(p^2+(p|4))A(n)-p^2(p|4)A(n/p),$$
$$B(pn)=(1+p^2(p|4))B(n)-p^2(p|4)B(n/p).$$

Therefore we can apply our theorem, and find that the coefficients of q^{m^2} in

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2,$$

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)\left\{-\frac{1}{4}+\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(-1)^n(2n+1)^2\right\},$$

are respectively equal to

$$\sum M_1\left(\frac{2m}{r}\right)r^2(r|4)\mu(r), \qquad \sum_r M_2\left(\frac{2m}{r}\right)r^2(r|4)\mu(r),$$

where

$$\sum_0^\infty M_1(n)q^n = \left\{\sum_1^\infty \frac{q^n}{1+q^{2n}}n^2\right\}^2,$$

$$\sum_0^\infty M_2(n)q^n = \left\{-\frac{1}{4}+\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}(2n+1)^2(-1)^n\right\}^2.$$

Thus, collecting terms, we find that the coefficient of q^{m^2} in

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots)^6$$

is equal to $\sum\left\{16M_1\left(\frac{2m}{r}\right)-4M_2\left(\frac{2m}{r}\right)\right\}r^2(r|4)\mu(r),$

where

$$\sum_0^\infty M_1(n)q^n = \frac{1}{256}\left(\frac{2K}{\pi}\right)^6 k^4, \qquad \sum_0^\infty M_2(n)q^n = \frac{1}{16}\left(\frac{2K}{\pi}\right)^6(1-k^2)^2.$$

To evaluate $\sum_0^\infty \{16M_1(2n)-4M_2(2n)\}q^{2n},$

we change q into $-q$ in the previous two formulae by changing

$$k^2 \text{ into } \frac{k^2}{k^2-1} \quad \text{and} \quad \frac{2K}{\pi} \text{ into } \sqrt{(1-k^2)}\frac{2K}{\pi},$$

and find

$$\sum_0^\infty M_1(n)(-q)^n = \frac{1}{256}\left(\frac{2K}{\pi}\right)^6 k^4(1-k^2),$$

$$\sum_0^\infty M_2(n)(-q)^n = \frac{1}{16}\left(\frac{2K}{\pi}\right)^6(1-k^2).$$

Hence, adding to the two previous formulae, we have

$$\sum_0^\infty M_1(2n)q^{2n} = \frac{1}{512}\left(\frac{2K}{\pi}\right)^6 k^4(2-k^2),$$

$$\sum^\infty M_2(2n)q^{2n} = \frac{1}{32}\left(\frac{2K}{\pi}\right)^6(1-k^2)(2-k^2).$$

Therefore

$$\sum_0^\infty \{16M_1(2n)-4M_2(2n)\}q^{2n} = -\frac{1}{32}\left(\frac{2K}{\pi}\right)^6(2-k^2)(4-4k^2-k^4)$$

$$= -\frac{1}{72}\left(\frac{2K}{\pi}\right)^6(1+k^2)(1-2k^2)(2-k^2) - \frac{1}{288}\left(\frac{2K}{\pi}\right)^6(2-k^2)(32-32k^2-k^4).$$

Equating coefficients of x^4 in the power-series expansion and the partial-fraction expansion of $1/\mathrm{sn}^2x$, we have

$$\frac{1}{2}\left(\frac{2K}{\pi}\right)^6(1+k^2)(1-2k^2)(2-k^2) = 1-504\sum_1^\infty \frac{q^{2n}}{1-q^{2n}}n^5.$$

Also, changing q into q^2 by changing

$$k \text{ into } \frac{1-\sqrt{(1-k^2)}}{1+\sqrt{(1-k^2)}} \quad \text{and} \quad \frac{2K}{\pi} \text{ into } \frac{1+\sqrt{(1-k^2)}}{2}\frac{2K}{\pi},$$

we have

$$\frac{1}{64}\left(\frac{2K}{\pi}\right)^6(2-k^2)(32-32k^2-k^4) = 1-504\sum_1^\infty \frac{q^{4n}}{1-q^{4n}}n^5.$$

Therefore

$$\sum_0^\infty \{16M_1(2n)-4M_2(2n)\}q^{2n}$$

$$= -\frac{1}{36}\left\{1-504\sum_1^\infty \frac{q^{2n}}{1-q^{2n}}n^5\right\} - \frac{2}{9}\left\{1-504\sum_1^\infty \frac{q^{4n}}{1-q^{4n}}n^5\right\}.$$

Hence $$16M_1(2n)-4M_2(2n) = 14\sigma_5(n)+112\sigma_5(n/2),$$

and therefore

$$r_7(m^2) = 14\sum_{r\,\text{odd}}\left\{\sigma_5\left(\frac{m}{r}\right)+8\sigma_5\left(\frac{m}{2r}\right)\right\}r^2(r|4)\mu(r),$$

as was to be proved.

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ON THE SUM OF A TERMINATING ${}_3F_2(1)$

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[Received 23 March 1953]

1. APART from Saalschütz's theorem, which sums a terminating ${}_3F_2$, there are three known cases in which a ${}_3F_2$ can be summed. These are

$$ {}_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b,\ 1+a-c \end{matrix}\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}, \tag{1.1} $$

$$ {}_3F_2\left[\begin{matrix} a, b, c; \\ \frac{1}{2}(a+b+1),\ 2c \end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)}, \tag{1.2} $$

$$ {}_3F_2\left[\begin{matrix} a, 1-a, c; \\ e,\ 1+2c-e \end{matrix}\right] = \frac{\pi\Gamma(e)\Gamma(1+2c-e)}{2^{2c-1}\Gamma(\frac{1}{2}a+\frac{1}{2}e)\Gamma(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}e)\Gamma(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}e+c)\Gamma(1-\frac{1}{2}a-\frac{1}{2}e+c)}, \tag{1.3} $$

due respectively to Dixon, Watson, and Whipple.†

When c in (1.1) is a negative integer $-m$, so that the series on the left terminates, the formula gives the sum as a finite product, and it is then valid for all values of the parameters (except those which make terms infinite). The same is true‡ for (1.1) when $a \to -m$ (m even), for (1.2) when $a = -m$ (m even), and for (1.3) when $a = -m$.

The situation is different in (1.2) and (1.3) when we attempt to let $c \to -m$. In (1.2) the resulting formula is correct, but the series on the left then consists of $m+1$ terms followed by a number of zero terms and then an infinite series. In (1.3) the condition for convergence is $R(c) > 0$, and we cannot justify putting $c = -m$. In fact, if we do this formally we get a result which is definitely incorrect except when a is also an integer.

There thus remain the two questions, whether we can (i) sum the series in (1.2) when $c = -m$ and the series terminates after $m+1$ terms, (ii) sum the series in (1.3) when $c = -m$.

† See W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge Tract, 1935), §§ 3.1, 3.3, 3.4.

‡ When a is a negative odd integer, the sums of the series on the left of (1.1) and (1.2) are, of course, both zero.

It will be shown that the answers to both these questions are in the affirmative, and that the result for (1.3) can be put in an interesting symmetrical form.

2. We start with the formula

$$ {}_3F_2\left[\begin{matrix}a, b, -m;\\ e, f\end{matrix}\right] = \frac{(f-b)_m}{(f)_m}\,{}_3F_2\left[\begin{matrix}e-a, b, -m;\\ e, 1+b-f-m\end{matrix}\right], \tag{2.1} $$

which is, of course, a known result. It can easily be deduced from the relation

$$ F[a, b; c; z] = (1-z)^{-b}F[c-a, b; c; -z/(1-z)] $$

by multiplying by $(1-z)^{f+m-1}$ and equating coefficients of z^m.

Now take $e = \frac{1}{2}(a+b+1)$, $f = -2m$ in (2.1) and we can sum the series on the right by (1.1) with $c = -m$. We get

$$ {}_3F_2\left[\begin{matrix}a, b, -m;\\ \frac{1}{2}(a+b+1), -2m\end{matrix}\right] = \frac{(\frac{1}{2}+\frac{1}{2}a)_m(\frac{1}{2}+\frac{1}{2}b)_m}{(\frac{1}{2})_m(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)_m}, \tag{2.2} $$

where the series on the left terminates after $m+1$ terms. The result can also be obtained by reversing the series in (2.2), when it becomes a series which can be summed by (1.3) with $a = -m$.

3. Now in (2.1) take $b = 1-a, f = 1-2m-e$, and we again get a ${}_3F_2$ on the right which can be summed by (1.1) with $c = -m$. The result is

$$ {}_3F_2\left[\begin{matrix}a, 1-a, -m;\\ e, 1-2m-e\end{matrix}\right] = \frac{2^{2m}(\frac{1}{2}+\frac{1}{2}e-\frac{1}{2}a)_m(\frac{1}{2}e+\frac{1}{2}a)_m}{(e)_{2m}}. \tag{3.1} $$

This can be written in the symmetrical form

$$ \sum_{r=0}^{m}(-1)^r\binom{m}{r}(a)_r(1-a)_r(b)_{m-r}(1-b)_{m-r} $$

$$ = (-1)^m 2^{2m}(\tfrac{1}{2}a+\tfrac{1}{2}b-\tfrac{1}{2}m)_m(\tfrac{1}{2}-\tfrac{1}{2}a+\tfrac{1}{2}b-\tfrac{1}{2}m)_m \tag{3.2} $$

when we replace e by $b-m$.

4. The formula (1.1) when $a \to -2m$ (and c is then changed to a) can be written in the symmetrical form

$$ \sum_{r=0}^{2m}(-1)^r\binom{2m}{r}(a)_r(b)_r(a)_{2m-r}(b)_{2m-r} = 2^{2m}(\tfrac{1}{2})_m(a)_m(b)_m(a+b+m)_m, \tag{4.1} $$

the sum being zero when $2m$ is replaced by an odd integer.

From (3.2) and (4.1) it is easily shown that

$$F_3[a,b;1-a,1-b;c;x,-x]$$
$$= {}_4F_3\left[\begin{matrix}\frac{1}{2}(a+b),1-\frac{1}{2}(a+b),\frac{1}{2}(1+a-b),\frac{1}{2}(1-a+b);\\ \frac{1}{2},\frac{1}{2}c,\frac{1}{2}(1+c)\end{matrix}\ x^2\right] -$$
$$-\frac{(a-b)(a+b-1)x}{c}\times$$
$$\times {}_4F_3\left[\begin{matrix}\frac{1}{2}(1+a+b),\frac{1}{2}(3-a-b),1+\frac{1}{2}(a-b),1-\frac{1}{2}(a-b);\\ \frac{3}{2},\frac{1}{2}(1+c),1+\frac{1}{2}c\end{matrix}\ x^2\right], \tag{4.2}$$

$$F_3[a,a;b,b;c;x,-x] = {}_4F_3\left[\begin{matrix}\frac{1}{2}(a+b),\frac{1}{2}(a+b+1),a,b;\\ a+b,\frac{1}{2}c,\frac{1}{2}(1+c)\end{matrix}\ x^2\right], \tag{4.3}$$

where F_3 is Appell's function of two variables.

Similarly, from (1.2) with $a = -m$, we obtain

$$F_2[a;b,c;2b,2c;x,-x] = {}_4F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}(a+1),\frac{1}{2}(b+c),\frac{1}{2}(b+c+1);\\ b+\frac{1}{2},c+\frac{1}{2},b+c\end{matrix}\ x^2\right], \tag{4.4}$$

and from (1.1) we get

$$F_2[a;b,b;c,c;x,-x] = {}_4F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}(a+1),b,c-b;\\ c,\frac{1}{2}c,\frac{1}{2}(c+1)\end{matrix}\ x^2\right], \tag{4.5}$$

$$F_2[a;b,1+b-c;c,2-c;x,-x]$$
$$= {}_4F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}(1+a),\frac{1}{2}(1+c)-b,\frac{1}{2}(1-c)+b;\\ \frac{1}{2},\frac{1}{2}(1+c),\frac{1}{2}(3-c)\end{matrix}\ x^2\right] -$$
$$-\frac{a(1-c)(c-2b)x}{c(2-c)}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}(1+a),1+\frac{1}{2}a,1-\frac{1}{2}c+b,1+\frac{1}{2}c-b;\\ \frac{3}{2},2-\frac{1}{2}c,1+\frac{1}{2}c\end{matrix}\ x^2\right] \tag{4.6}$$

The formulae (4.3) and (4.5) are reminiscent of expansions given by Burchnall and Chaundy,† who gave formulae involving the series on the left of (4.3) and (4.5) with general variables x, y.

A formula corresponding to (4.4), which I have not seen elsewhere, is

$$F_2[a;b,c;2b,2c;2x,2y]$$
$$= (1-x-y)^{-a}F_4\left[\tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2};b+\tfrac{1}{2},c+\tfrac{1}{2};\frac{x^2}{(1-x-y)^2},\frac{y^2}{(1-x-y)^2}\right]. \tag{4.7}$$

† J. L. Burchnall and T. W. Chaundy, *Quart. J. of Math.* (Oxford) 11 (1940), 249–70, (49) and (43).

This can be proved by twice using Kummer's formula

$$F[a, b; 2b; 2x] = (1-x)^{-a} F\left[\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; b+\tfrac{1}{2}; \frac{x^2}{(1-x)^2}\right].$$

It can also be verified by comparing the coefficients of $x^m y^n$ on the two sides, and twice using the formula

$$F[-\tfrac{1}{2}m, \tfrac{1}{2}-\tfrac{1}{2}m; b+\tfrac{1}{2}] = 2^m (b)_m/(2b)_m,$$

which is a particular case of Gauss's theorem.

The formula (4.7) can be extended to functions with any number of variables. For example

$$F_A[\alpha; \beta_1, \beta_2, \beta_3; 2\beta_1, 2\beta_2, 2\beta_3; 2x, 2y, 2z]$$
$$= (1-x-y-z)^{-\alpha} F_C\left[\tfrac{1}{2}\alpha, \tfrac{1}{2}\alpha+\tfrac{1}{2}; \beta_1+\tfrac{1}{2}, \beta_2+\tfrac{1}{2}, \beta_3+\tfrac{1}{2};\right.$$
$$\left.\frac{x^2}{(1-x-y-z)^2}, \frac{y^2}{(1-x-y-z)^2}, \frac{z^2}{(1-x-y-z)^2}\right], \tag{4.8}$$

with the usual notation for functions of three variables.

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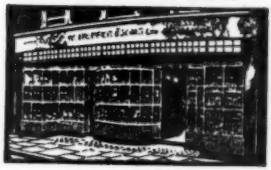

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